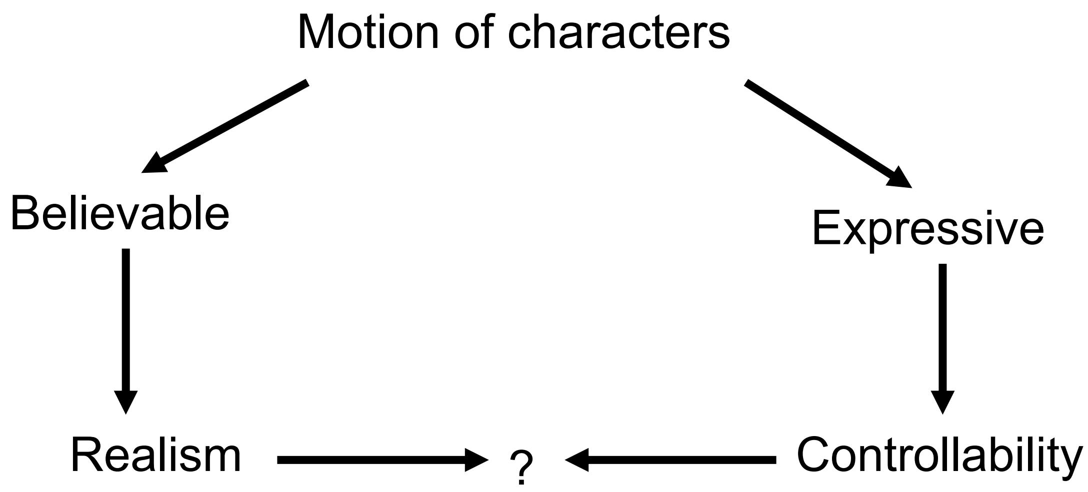
Kinematical Animation

lionel.reveret@inria.fr

3D animation in CG

- Goal : capture visual attention

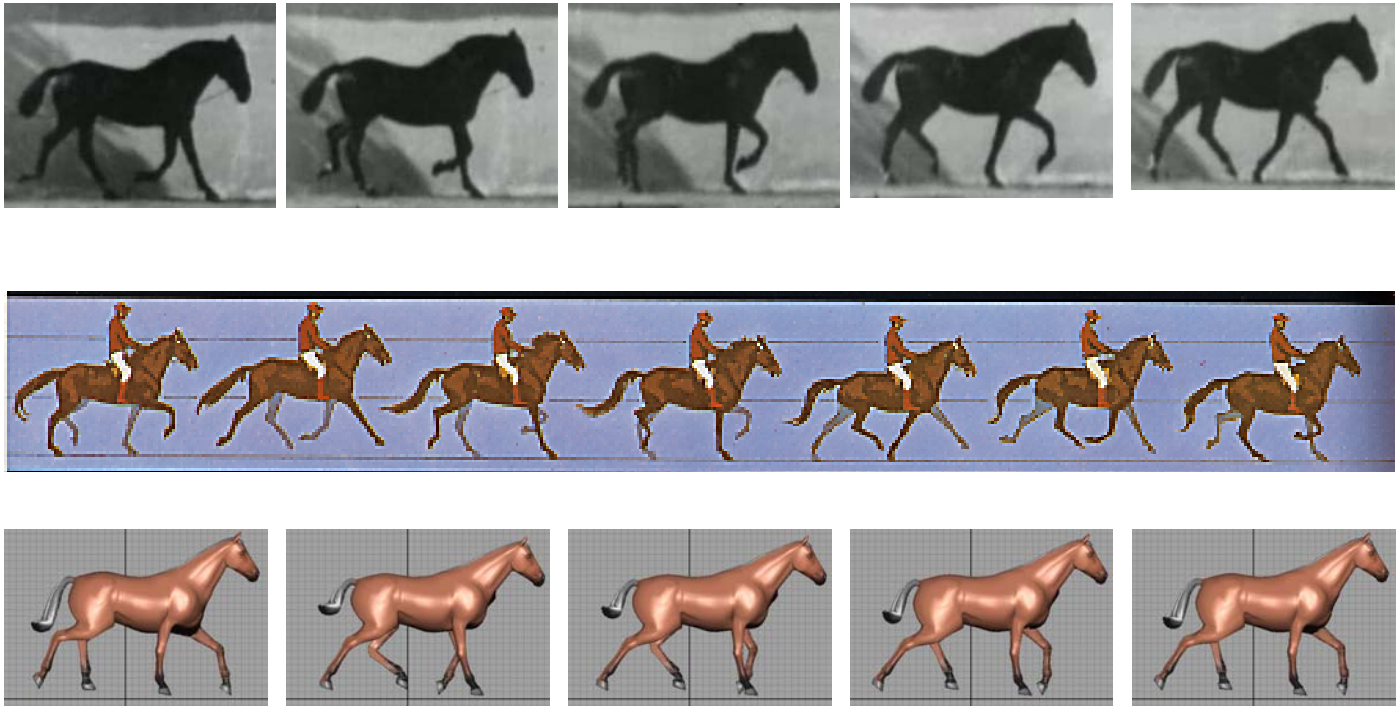


Limits of purely physical simulation :
- little interactivity
- high complexity for expressive characters

animation in 3D CG

- Two heritages
  - Cartoons
    - How to represent motion on a 2D visualization
  - Robotics
    - Mathematics foundation of 3D motion

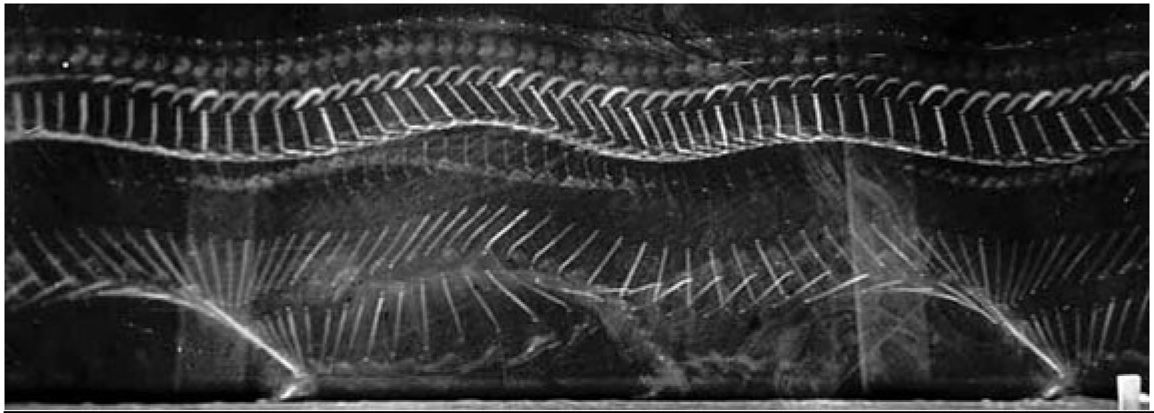
Heritage from cartoon



- Sampled motion, film framework (24 images per second)
- Animation workflow from photographs (Muybridge, Marey)

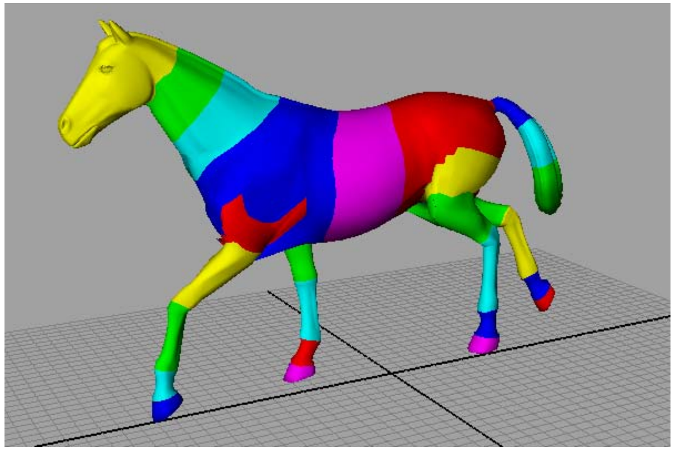
Early Mathematics of motion

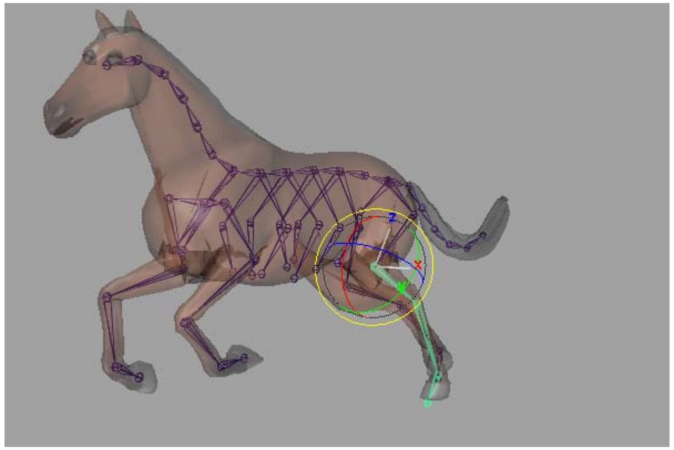
- Etienne-Jules Marey (1830-1904)
  - physiologist
  - inventor of chronophotography (1882)
    - cinematography invented in 1895 (L. Lumière)



The “graphical method”:
one motion can be represented by a curve

Heritage from robotics





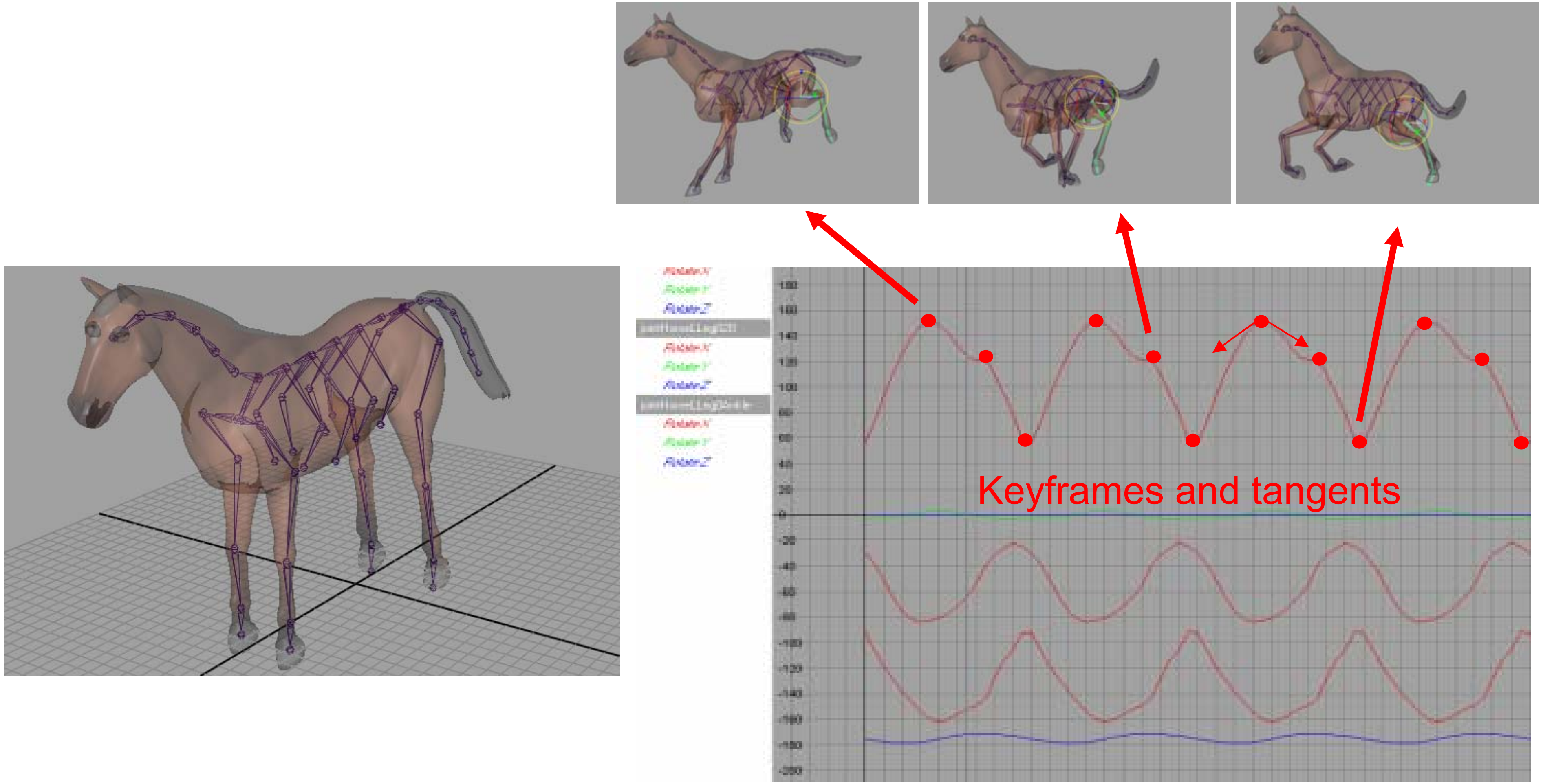
Chain of rigid articulations

+ Forward kinematics
+ Inverse-Kinematics algorithm
+ Motion planning

Animation skeleton :

appropriate degrees of freedom for expressivity

3D animation : interpolation+skeleton

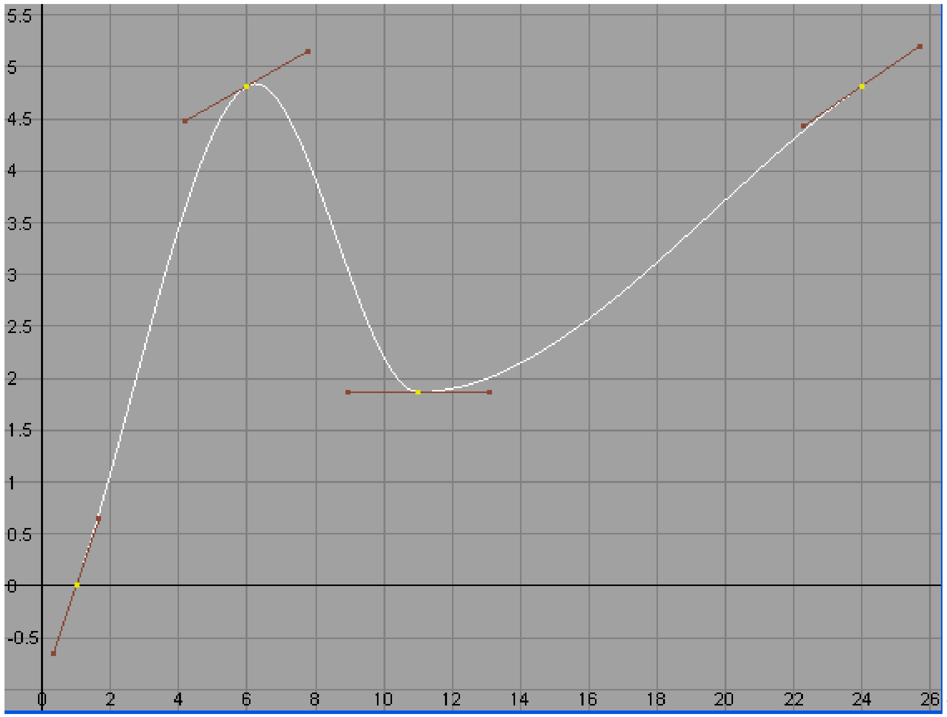


Interpolation

- Interpolation of a 1D-scalar value
  - One function p(u) with unknown parameters,
    - Typically cubic polynomial: $p(u)=\Sigma_{n=0..3}\ a_n\ u^n$
  - $C^{(n)}$ known constrains on points
    - $\{\ u_i\ ,\ (d^n p/du)(u_i\ )=\ b_i\ \}$
    - 4 are enough for exact cubic polynomial
  - Direct solving for a_n
    - p(u) is thus known for every u

Interpolation

- Practical case : Hermit polynomial (spline)
  - C^1 continuity between sets of 2 points
    - 2 positions and tangents at these positions



$$p(u) = [\ a_0\ a_1\ a_2\ a_3\ ]\ [\ 1\ u\ u^2\ u^3\ ]^t = A^t\, Q(u)$$

$$\begin{aligned} p(0) &= A^t\, Q(0) = b_0 \\ p'(0) &= A^t\, Q'(0) = b_1 \\ p(1) &= A^t\, Q(1) = b_2 \\ p'(1) &= A^t\, Q'(1) = b_3 \end{aligned}$$

$$A^t\, [\ Q(0)\ Q'(0)\ Q(1)\ Q'(1)\ ] = [\ b_0\ b_1\ b_2\ b_3\ ]^t$$

$$A^t\, Q_{4x4} = B^t \quad \Rightarrow \quad A^t = B^t\, Q^{-1}$$

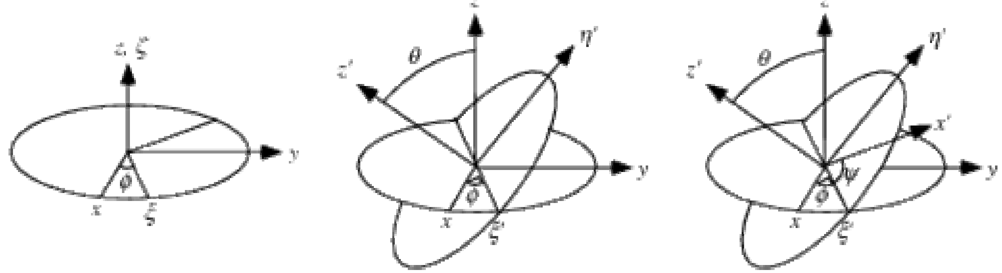
$$p(u) = B^t\, Q^{-1}\, Q(u)$$

$$p(\ u=(t-t0)/(t-t1)\ )$$

Interpolation

- Interpolating 3D rotation
  - Canonic representation : SO(3) matrix
    - but $M_0, M_1 \in SO(3) \not\Rightarrow (1-\alpha)M_0 + \alpha M_1 \notin SO(3)$
  - Represent SO(3) matrix with Euler angles
    - any rotation in R^3 can be represented by 3 angles
    - $M = R_{x,\psi} R_{y,\theta} R_{z,\varphi} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\psi & \sin\psi \\ 0 & -\sin\psi & \cos\psi \end{bmatrix} \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Multiplication order matters !



=> Animation by interpolating angles

Interpolating 3D rotation

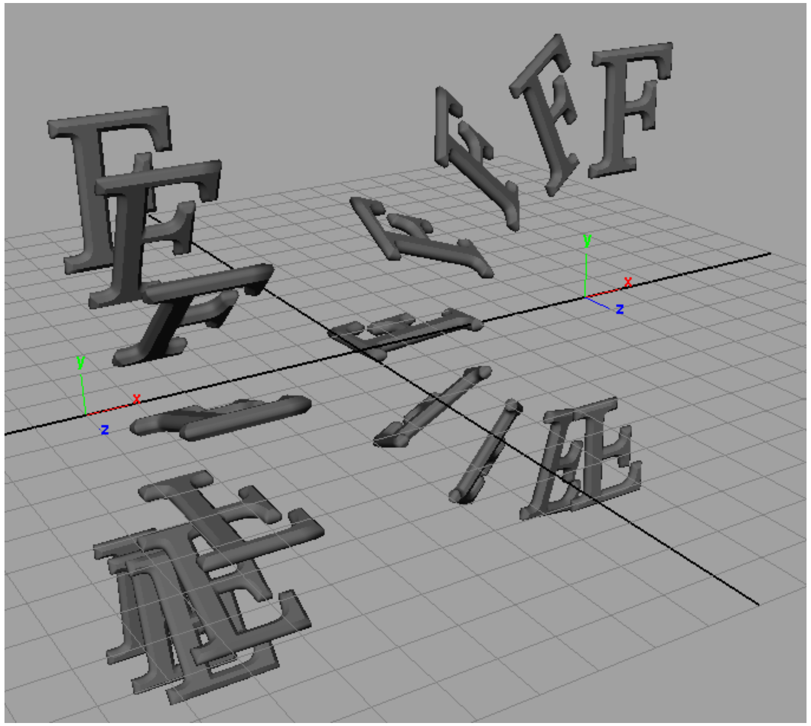
- Limitations of Euler angles
  - Non-uniqueness of position and path

$[\theta_x,\theta_y,\theta_z]=[0,0,0]$

↓

$[\theta_x,\theta_y,\theta_z]=[\pi,0,0]$

$$M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$



$[\theta_x,\theta_y,\theta_z]=[0,0,0]$

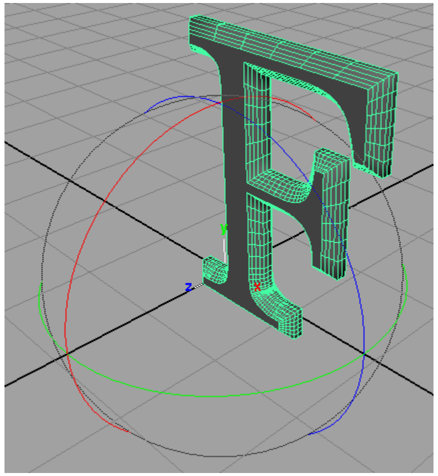
↓

$[\theta_x,\theta_y,\theta_z]=[0,\pi,\pi]$

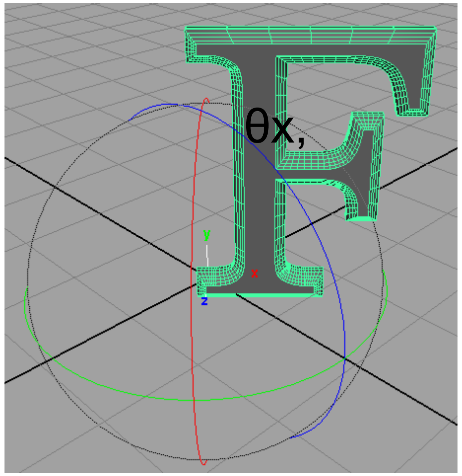
$$M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Interpolating 3D rotation

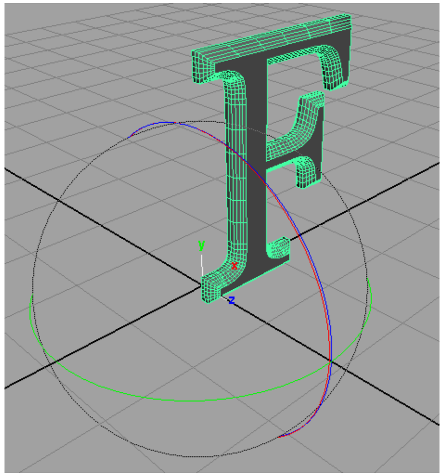
- Limitations of Euler angles
  - Gimbal lock



$[\theta_x,\theta_y,\theta_z]=[0,0,0]$



$[\theta_x,\theta_y,\theta_z]=[0,\pi/4,0]$

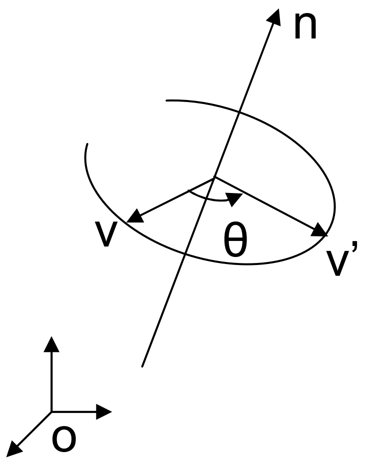


$[\theta_x,\theta_y,\theta_z]=[0,\pi/2,0]$

1 degree of freedom is lost : change in $\theta_x \Leftrightarrow$ change in θ_z

3D rotation

- Axis-angle
  - any rotation in R^3 is a planar rotation around an axis
  - strong link with quaternion



$$v' = R_{\theta,n} v = \cos\theta\, v + \sin\theta\, n \times v + (1 - \cos\theta)(v \cdot n)\, n$$

$$R_{\theta,n} = \cos\theta\, Id + \sin\theta\, [n]_\times + (1 - \cos\theta) n n^t$$

Rodrigues formula

Quaternion

H: Extension of standard complex

$q = [\ s, x, y, z\ ] = s + \mathbf{i}x + \mathbf{j}y + \mathbf{k}z$

with $\mathbf{i}^2=\mathbf{j}^2=\mathbf{k}^2=\mathbf{ijk}=-1$

$q^* = [s, -x, -y, -z\ ]$

$||q||^2 = qq^* = s^2 + x^2 + y^2 + z^2$

$q^{-1} = q^*/||q||^2$

$||q||=1 \Rightarrow q = [\ \cos\theta, (\sin\theta)\mathbf{n}\ ] \in H_1$

with $\mathbf{n} \in R^3$ and $||\mathbf{n}||=1$

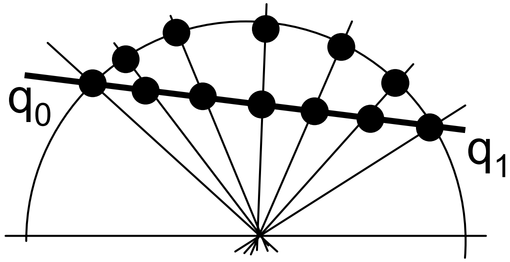
Any rotation in R^3 can be represented in H_1

$x \in R^3$, $x' = R_{\theta,n}x \Leftrightarrow [0,x'] = q[0,x]q^{-1}$

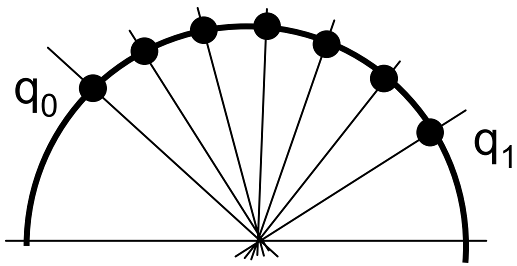
with $q = [\ \cos(\theta/2), \sin(\theta/2)\mathbf{n}\ ]$

Interpolating 3D rotation

- Quaternion
  - Linear interpolation in H does not work well
    - $q(t) = (1-t)*q_0 + t*q_1$
    - Angular velocity is not constant



  - Spherical linear interpolation is fine (SLERP)



Interpolating 3D rotation

Log and exp in H_1

- $q = [\cos\theta, (\sin\theta)\mathbf{n}] = \exp(\theta\mathbf{n})$
- $\log(q) = [0, \theta\mathbf{n}]$
- $q^t = \exp(t \log(q))$
- $dq^t/dt = \log q \; q^t$
- $\| dq^t/dt \| = \|\log q\| = \| [0, \theta\mathbf{n}] \| = |\theta|$

Application to SLERP

- $SLERP(q0,q1,t) = q_0(q_0^{-1}q_1)^t$
- $SLERP(q0,q1,t) = (\sin(\Omega - \Omega t)q_0 + \sin(\Omega t)q_1)/\sin\Omega$
  - with $\cos\Omega = q_0 \cdot q_1$

Kinematic Animation

- Two fundamental cases
  - Unconstrained motion
    - waving arms, nodding head, etc

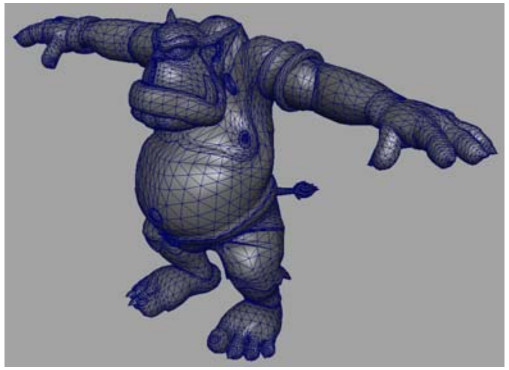
    => Forward Kinematics (FK)

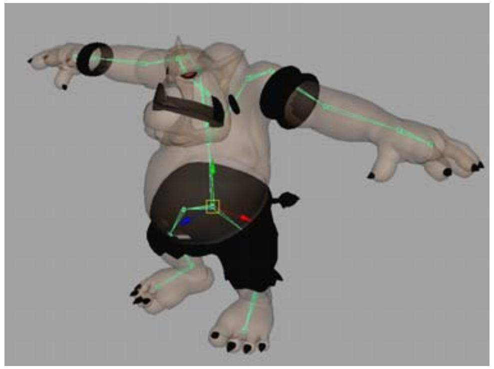
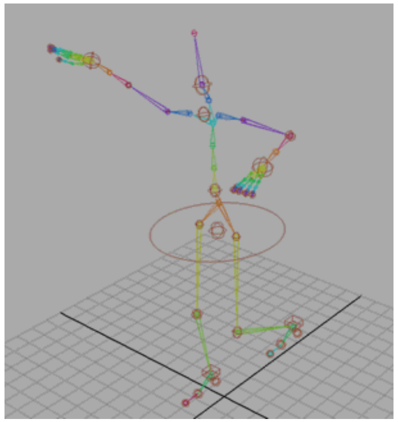
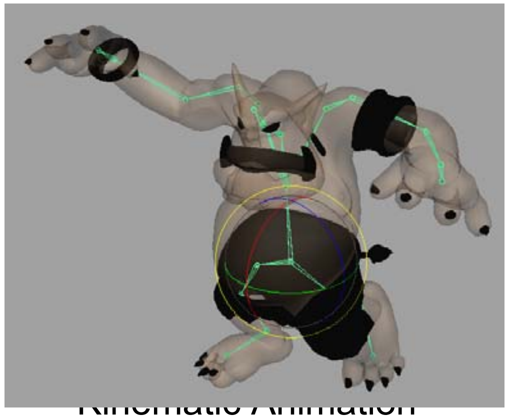
  - Constrained motion
    - grasping an object, walking on the ground, etc

    => Inverse Kinematics (IK)

Forward Kinematics

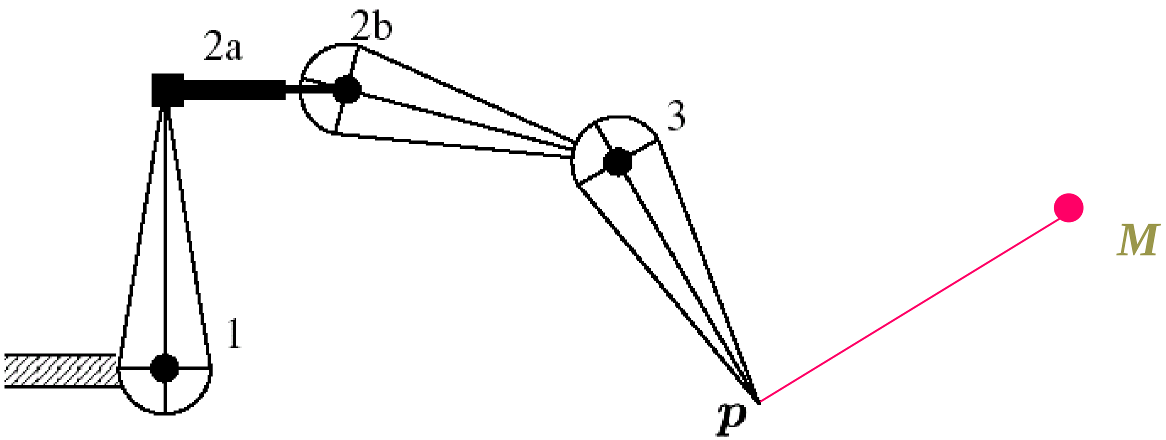
- Direct application of 3D framework




Inverse Kinematics

- Articulated object
  - Translational and rotational links
  - Goal to reach



Inverse Kinematics

- input : goal to reach (M)
- model parameter :

  $\Theta = (\theta_1, \theta_2, \ldots, t_1, t_2, \ldots)$, model parameters

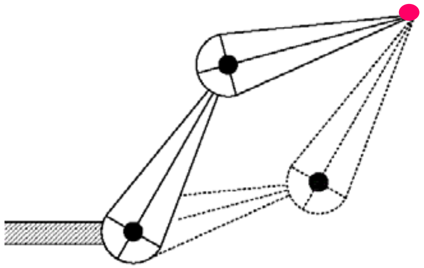
  $f(\Theta)$ position of kinematic chain end

$\Rightarrow$ Find $\Theta^* / M = f(\Theta^*)$

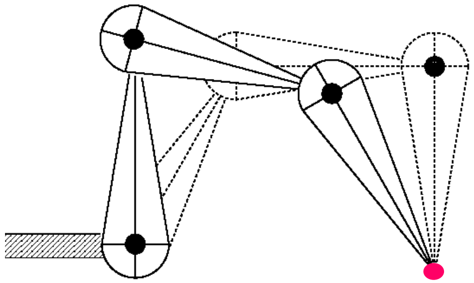
- 2 or 3 rotations: direct computation in R^2 or R^3
- N articulations : ?

Difficulties

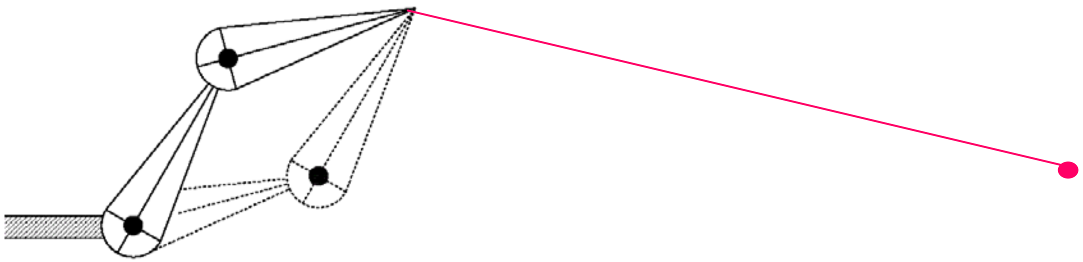
- Two solutions :



- Range of solutions :



- No solutions :



f : direct kinematics

- Concatenation of matrix transforms
- $f(\Theta) = \mathbf{R}_1(\theta_1)\mathbf{T}_1(t_1)\mathbf{R}_2(\theta_2)\mathbf{T}_2(t_2)\ldots M_0$
  - M_0 : position in rest pose (no rotations)
- Non linearity because of rotations

Zero of non-linear function

- Find Θ / $f(\Theta)$-M = 0

- Linearization :
  - given a current Θ and error E = $f(\Theta)$-M
  - find h / E = $f(\Theta+h) - f(\Theta) = f'(\Theta)h$

  => $h = f'(\Theta)^{-1} E$

  => $\Theta := \Theta + h$
  - iterate

Linearization

- Taylor series :

$$f(\Theta + h) = f(\Theta) + f'(\Theta)h + f''(\Theta)h^2 + \ldots$$

- Multivariate case :

$$f(\Theta + h) = f(\Theta) + \mathbf{J}(\Theta)h + {}^t h\mathbf{H}(\Theta)h + \ldots$$

- **J** Jacobian of *f*, linear from in h
- **H** Hessian of *f*, quadratic form in h

Jacobian

- Matrix of derivatives of several functions with severable variables :

$$J_{ij} = \frac{\partial f_i}{\partial x_j}$$

J:3xN matrix => not squared
Use Pseudo-inverse for inversion :

$J^+ = J^t(JJ^t)^{-1}$ if N>3

or $J^+ = (J^tJ)^{-1}J^t$ if N<3

Algorithm

```
inverseKinematics()
  {
    start with current Θ;
     E := target - computeEndPoint();
    for(k=0; k<k_max && |E| > eps; k++){
      J := computeJacobian();
      solve J h = E;
      Θ := Θ + h;
      E := target - computeEndPoint();
    }
  }
```

Joint limits

- Joint may have limits of variation
  - For example realistic elbow is limited

- To enforce limits :
  - test for limitation violation
  - cancel parameter if violation
    - in practice, remove column in J
  - compute new J and J^+
  - compute new h

Adding constrains

1. if $\mathrm{KerJ} \neq 0$, degrees of freedom left to enforce a new constrains Ω

   $\mathbf{J}\Omega = 0$

2. if Θ solves $\mathbf{J}\Theta = E$, thus $\Theta+\Omega$ is also solution

   $\mathbf{J}(\Theta+\Omega) = \mathbf{J}\Theta + \mathbf{J}\Omega = E + 0 = E$

3. if general constrain C, need to project on KerJ:

   $C_p = (J^+J - I)\, C$

   check: $J\, C_p = J\,(J^+J - I)\, C = (J - J)\, C = 0$

lionel.reveret@inria.fr

Example: preferred angle

- Value : Θ_{pref}
- Constraint C :
  - $C_i = \Theta_i - \Theta_{pref}$

- Modified algorithm :
  - use $\mathbf{h} = \mathbf{J}^{+}E + (\mathbf{J}^{+}\mathbf{J} - \mathbf{I})C$

  => preserve convergence

Inverse Kinematics

- Other methods
  - use J^t instead of J^+
    - theory of infinitesimal works
    - $h = J^t E$
  - use several 1D optimization
    - Cyclic Coordinate Descent

=> faster but less accurate

lionel.reveret@inria.fr