Mathematical tools 1
Session 1

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M2R IVR, October 5th 2006

Organization

- Talk in French, slides in English
  $\rightsquigarrow$ *some technical phrases will be given both in French and in English*
- 8 sessions (1.5 h each)
- Written exam (2h), handwritten documents allowed
- Feel free to ask questions or make (relevant) comments whenever you want
- My e-mail address: Franck.Hetroy@imag.fr
- Webpage (slides, planning, bibliography, . . .):
  http://www-evasion.imag.fr/Membres/Franck.Hetroy/Teaching/OutilsMaths1/

Outline of the course

Two parts:

1. Interpolation and approximation (4 sessions)
   1. Splines (2 sessions)
   2. Wavelets and multiresolution (2 sessions)
2. Position, orientation and motion (4 sessions)
   1. Representations of position, orientation, and rotations (2 sessions)
   2. Kinematics (2 sessions)

Prerequisite

Basic knowledge in maths:

- Affine spaces ($\mathbb{R}^2$, $\mathbb{R}^3$), points and vectors
- Polynomials
- Derivates, continuity, (piecewise) continuous functions
- Curve $\neq$ function

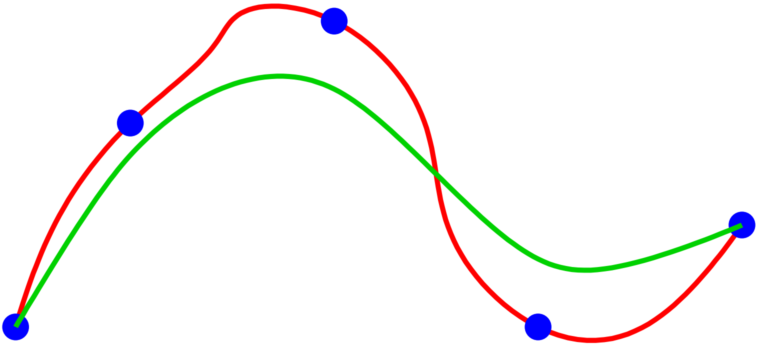
$\rightsquigarrow$ *If you have any problem, please ask me*

Interpolation and approximation

1. Motivations
2. Splines
3. Wavelets and multiresolution

Interpolation and approximation

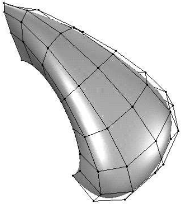
Interpolation vs. approximation



Motivations

Discrete data to process

- Compute paths for robots or cameras going through or near some known keypoints
- Define shapes (curves, surfaces) starting from 2D or 3D points (ex.: car industry)
- Find clever intermediate values for some parameters



[G. Farin]

Interpolation and approximation

Interpolation and approximation

Spline curve

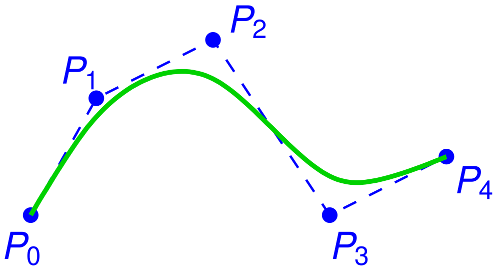
Definition (Spline curve, control points and network, influence functions)

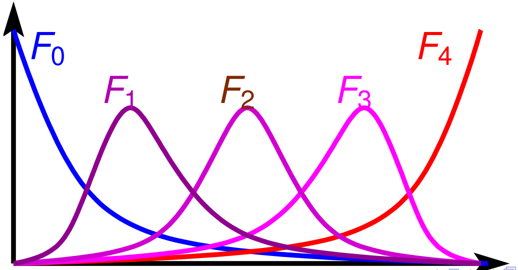
Let $\mathcal{P} = \{P_1, \ldots, P_k\}$ be a set of k points in the plane. Let $\mathcal{F} = \{F_1, \ldots, F_k\}$ be a set of k functions defined on $[0, 1]$ to $\mathbb{R}$. We call spline curve generated by the couples (P_i, F_i), $1 \leq i \leq k$, the curve C which parametric equation is:

$$\forall t \in [0, 1], C(t) = \sum_{i=1}^{k} F_i(t) P_i \tag{1}$$

Points P_i are called control points of C.
Functions F_i are called influence functions of C.
The ordered set of points P_i and segments $P_i P_{i+1}$ is called the control network of C.

Example





Our goal

Find influence functions F_i:

- such that a spline curve has interesting properties
- not too complicated or computationally costly

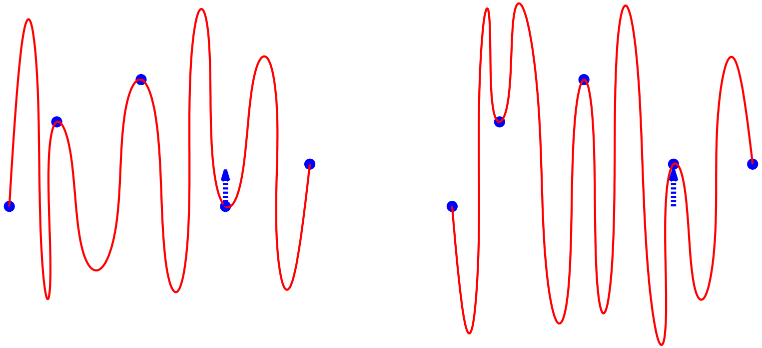
First idea

Lagrange polynomials:

$$C(t) = \sum_i P_i \prod_{j \neq i} \frac{t - t_j}{t_i - t_j}$$

+ interpolation
– not so easy to compute
– a lot of unwanted oscillations
– if we move one point, the curve is modified everywhere

Lagrange polynomials



Interesting properties

We would like to:

1. know how a spline curve will be transformed into another when the control network is modified (in a homogeneous way);
2. get spline curves as “simple” or “natural” as possible (should not have many oscillations);
3. modify a spline curve only locally when moving a control point;
4. control the continuity of a spline curve.

Interesting properties

We would like to:

1. know how a spline curve will be transformed into another when the control network is modified (in a homogeneous way);

Interesting properties, 1/4

Definition (Normality)

A spline curve C is normal if

$$\forall t \in [0,1], \sum_{i=0}^{k} F_i(t) = 1 \tag{2}$$

Property (Affine invariance)

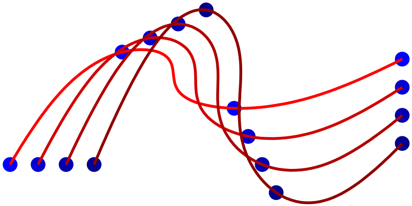
The affine transform of a normal spline curve can be computed simply by applying the affine transformation to its control points.

Property (Barycentric invariance)

The weighted mean of several normal spline curves (using the same influence functions) can be computed simply by computing the weighted mean of their control points.

Consequences

- The shape of a normal spline curve is independent of the frame in which its control points are expressed.
- If we translate, rotate or apply an homothecy to the control points of a normal spline curve, the curve is moved similarly and its shape is not modified.
- A normal spline curve can be continuously morphed into another normal spline curve, simply by interpolating their control points.



Interesting properties

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Interesting properties

We would like to:

2. get spline curves as “simple” or “natural” as possible (should not have many oscillations);

Interesting properties, 2/4

Definition (Positivity)

A spline curve C is positive if

$$\forall t \in [0,1], \forall i = 1 \dots k, F_i(t) \geq 0 \tag{3}$$

Definition (Regularity)

A spline curve C is regular if all $F_i, 1 \leq i \leq k$, are differentiable and

$$\forall i = 1 \dots k, \exists T_i \in [0,1] / \tag{4}$$

$$\forall t < T_i, F_i'(t) \geq 0 \text{ and } \forall i' = i+1 \dots k, F_{i'}(t) \leq F_i(t)$$

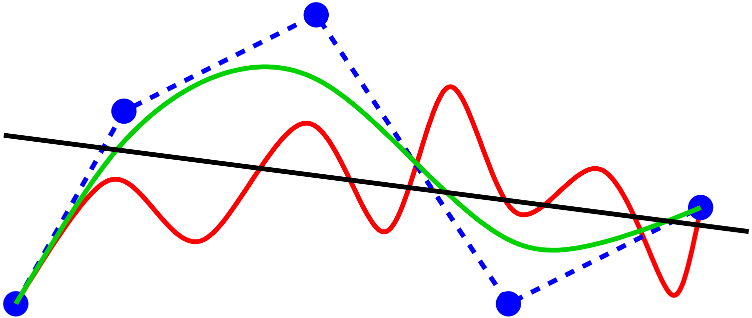
$$\forall t > T_i, F_i'(t) \leq 0 \text{ and } \forall i' = 0 \dots i-1, F_{i'}(t) \leq F_i(t)$$

This means that each F_i has a single maximum ($t = T_i$), and that F_i intersects F_j only between T_i and T_j.

Interesting properties, 2/4

Property (Oscillation regularization)

The number of intersections between a straight line and a normal, positive and regular spline curve is at most the number of intersections between this line and the curve's control network.



Interesting properties

We would like to:

1. know how a spline curve will be transformed into another when the control network is modified (in a homogeneous way);
2. get spline curves as “simple” or “natural” as possible (should not have many oscillations);
3. modify a spline curve only locally when moving a control point;
4. control the continuity of a spline curve.

Interesting properties

We would like to:

3. modify a spline curve only locally when moving a control point;

Interesting properties, 3/4

Definition (Locality)

A spline curve C is local if

$$\forall i = 1 \dots k, \exists (T_i^-, T_i^+) \in [0,1]^2 \text{ such that}$$
$$\forall t < T_i^-, F_i(t) = 0 \text{ and } \forall t > T_i^+, F_i(t) = 0. \quad (5)$$

This means that each F_i, thus each P_i, affects $C(t)$ only for $t \in [T_i^-, T_i^+]$.

Corollary: $\forall t$, the number of control points affecting $C(t)$ is reduced: usually, $C(t) = \sum_{i=i_0}^{i_1} F_i(t) P_i$.

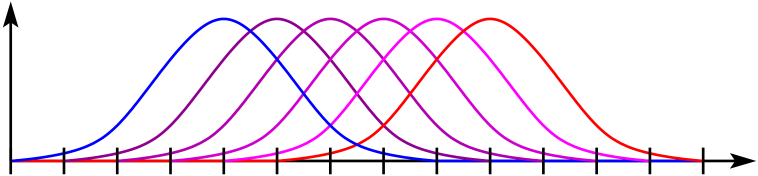
Interesting properties, 3/4

Definition (Curve segment)

The set of all t for which the same control points affect $C(t)$ is called a curve segment. It is the intersection of several intervals $[T_i^-, T_i^+]$.

Definition (Order of locality)

A spline curve C is local of order m if each control point affects at most m curve segments.



What is the order of locality here ?

Interesting properties

We would like to:

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4. control the continuity of a spline curve.

Interesting properties

We would like to:

4. control the continuity of a spline curve.

Interesting properties, 4/4

Two ways to define the smoothness of a curve:

Definition (Parametric continuity)

A spline curve C has a parametric continuity of order n (noted as C^n) if the function $t \mapsto \frac{d^n C}{dt^n}(t)$ is defined and continuous on $[0, 1]$.

Definition (Geometric continuity)

A spline curve C has a geometric continuity of order n (noted as G^n) if the function $s \mapsto \frac{d^n C}{ds^n}(s)$ is defined and continuous on $[0, 1]$ (s is the arc length of the curve).

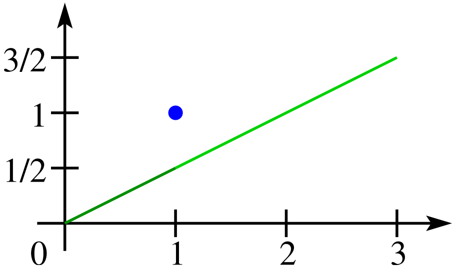
These definitions are important for local splines, since there are made of several successive segments, meeting in some points.

Parametric vs. geometric continuity

- Which is most interesting ?
  - Parametric continuity:
    + easy to compute
    – dependent on the parametrization
  - Geometric continuity:
    + independent on the parametrization
    – not always easy to compute
- Characterization:
  - C is G^1-continuous if $\exists k_1/\forall t, C$ is differentiable in t and $\frac{dC}{dt}(t^-) = k_1.\frac{dC}{dt}(t^+)$
  - C is G^2-continuous if $\exists k_1, k_2/\forall t, \frac{dC}{dt}$ is differentiable in t, $\frac{dC}{dt}(t^-) = k_1.\frac{dC}{dt}(t^+)$ and $\frac{d^2C}{dt^2}(t^-) = k_2.\frac{d^2C}{dt^2}(t^+)$
- $G^1 < C^1, G^2 < C^2, \ldots$

Example

$$C(t) = F_1(t).P_1,\; P_1 = (1,1)$$
$$F_1(t) = (2t, t) \text{ if } 0 \leq t \leq 1/2$$
$$F_1(t) = (4t-1, 2t-1/2) \text{ if } 1/2 < t \leq 1$$



$\rightsquigarrow$ C is G^1 but not C^1 !

$\rightsquigarrow$ path is a straight line but speed is not constant

Interesting properties

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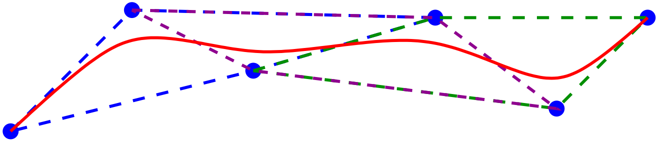
Other properties

Property (Convex hull)

A normal and positive spline curve is completely inside the convex hull of its control points.

Property (Strict convex hull)

A normal, positive, regular and local spline curve is completely inside the union of the convex hulls of the control points of its segments.



See you next week

The end !

Interpolation and approximation

Interpolation and approximation