Creating and processing 3D geometry

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Planning (provisional)

Part I – Geometry representations

- **Lecture 1 – Oct 9th – FH**
  - Introduction to the lectures; point sets, meshes, discrete geometry.

- **Lecture 2 – Oct 16th – MPC**
  - Parametric curves and surfaces; subdivision surfaces.

- **Lecture 3 – Oct 23rd - MPC**
  - Implicit surfaces.
Planning (provisional)

Part II – Geometry processing

- **Lecture 4 – Nov 6th – FH**
  - Discrete differential geometry; mesh smoothing and simplification *(paper presentations)*.

- **Lecture 5 – Nov 13th - CG + FH**
  - Mesh parameterization; point set filtering and simplification.

- **Lecture 6 – Nov 20th - FH (1h30)**
  - Surface reconstruction.
Planning (provisional)

Part III – Interactive modeling

• Lecture 6 – Nov 20th – MPC (1h30)
  − Interactive modeling techniques.
• Lecture 7 – Dec 04th - MPC
  − Deformations; virtual sculpting.
• Lecture 8 – Dec 11th - MPC
  − Sketching; paper presentations.
Discrete differential geometry

- Discrete surface: not smooth
- **Assumption:**
  \[ \text{mesh} = \text{piecewise linear approx. of a (real) smooth surface} \]
- **Goal:**
  find approximations of the differential properties of the underlying smooth surface
Applications

- Segmentation
- Remeshing
- Denoising or smoothing
- ...

Courtesy P. Alliez

Courtesy M. Meyer
Books

- **M. Botsch et al.**, “Geometric Modeling Based on Polygonal Meshes”, SIGGRAPH 2007 Course Notes, chapters 5 and 6.
  
  http://graphics.ethz.ch/~mbotsch/publications/sg07-course.pdf
  http://graphics.ethz.ch/~mbotsch/publications/meshcourse07_code.tgz

  
  http://ddg.cs.columbia.edu/
Today's planning

1. Differential geometry reminder
2. Discrete curvatures
3. Ridges and ravines
4. Other topics
5. Paper presentations
Differential geometry of a smooth curve

- $\gamma$ smooth parametric curve: $\gamma : I \rightarrow \mathbb{R}^n$
- Frenet vectors/frame:

\[
e_1(t) = \frac{\gamma'(t)}{\|\gamma'(t)\|}
\]

\[
e_j(t) = \frac{\overline{e}_j(t)}{\|\overline{e}_j(t)\|}, \quad \overline{e}_j(t) = \gamma^{(j)}(t) - \sum_{i=1}^{j-1} \langle \gamma^{(j)}(t), e_i(t) \rangle e_i(t)
\]

- The first 3 vectors are called the tangent, normal and binormal vectors
- If $n=3$, $e_3(t) = e_2(t) \times e_1(t)$
Curvature of a smooth curve

- Generalized curvature:
  \[ \chi_i(t) = \frac{\langle e'_i(t), e_{i+1}(t) \rangle}{\|\gamma'(t)\|} \]

- Curvature:
  \[ \kappa(t) = \chi_1(t) = \frac{\langle e'_1(t), e_2(t) \rangle}{\|\gamma'(t)\|} \]
  - Deviance from being a straight line

- Torsion:
  \[ \tau(t) = \chi_2(t) = \frac{\langle e'_2(t), e_3(t) \rangle}{\|\gamma'(t)\|} \]
  - Deviance from being a plane curve
Curvature of a plane curve

- $r = \text{curvature radius at } P = \frac{1}{\kappa(P)}$
- Osculating circle
- Exercise: $\gamma(t) = (K.\cos(t), K.\sin(t))$, $r = \ ?$
Differential geometry of a smooth surface

• S smooth parametric surface:

\[ x(u, v) = \begin{pmatrix} x(u, v) \\ y(u, v) \\ z(u, v) \end{pmatrix}, \quad (u, v) \in \mathbb{R}^2. \]

• Partial derivatives noted \( x_u \) and \( x_v \)

• Second partial derivatives noted \( x_{uu} \) etc.

• Normal vector: \( \mathbf{n} = (x_u \times x_v)/\|x_u \times x_v\| \)
Fundamental forms

- First fundamental form:

\[ I = \begin{bmatrix} E & F \\ F & G \end{bmatrix} := \begin{bmatrix} x_u^T x_u & x_u^T x_v \\ x_u^T x_v & x_v^T x_v \end{bmatrix} \]

- Second fundamental form:

\[ II = \begin{bmatrix} e & f \\ f & g \end{bmatrix} := \begin{bmatrix} x_{uu}^T n & x_{uv}^T n \\ x_{uv}^T n & x_{vv}^T n \end{bmatrix} \]

- Weingarten map/Shape operator:

\[ W := \frac{1}{EG - F^2} \begin{bmatrix} eG - fF & fG - gF \\ fE - eF & gE - fF \end{bmatrix} \]
Curvatures

- Principal directions and principal curvatures:

\[ W = \begin{bmatrix} \bar{t}_1 & \bar{t}_2 \end{bmatrix} \begin{bmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{bmatrix} \begin{bmatrix} \bar{t}_1 & \bar{t}_2 \end{bmatrix}^{-1} \]

- Mean curvature:

\[ H = \frac{\kappa_1 + \kappa_2}{2} = \frac{1}{2}\text{trace}(W) \]

- Gaussian curvature:

\[ K = \kappa_1 \kappa_2 = \text{det}(W) \]

**Fig. 5.** Curvature plots of a triangulated saddle using pseudo-colors: (a) Mean, (b) Gaussian, (c) Minimum, (d) Maximum. (Courtesy M. Meyer)
Example: torus

\[
\begin{align*}
\{ x[u, v] \\ y[u, v] \\ z[u, v] \} &= \begin{pmatrix}
\cos[u] (a + b \cos[v]) \\
(a + b \cos[v]) \sin[u] \\
b \sin[v]
\end{pmatrix}
\end{align*}
\]

Exercise:
Compute the Gaussian and mean curvatures.
Example: torus

\[ K[u, v] = \frac{\cos[v]}{b (a + b \cos[v])} \]

\[ H[u, v] = -\frac{a + 2b \cos[v]}{2b (a + b \cos[v])} \]
Laplace operator

- Laplace operator in an Euclidean space:
  \[ \Delta f = \text{div} \nabla f = \sum_i \frac{\partial^2 f}{\partial x_i^2} \]

- Laplace-Beltrami operator for (smooth) manifold surfaces:
  \[ \Delta_S f = \text{div}_S \nabla_S f \]

- Replacing f by the coordinates function:
  \[ \Delta_S x = -2H n \]
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Discrete differential geometry

- Discrete surface (here: mesh): not smooth
- **Goal:** find approximations of the differential properties of the underlying smooth surface
- **Idea:** express them as *averages* over a local neighborhood
Discrete Laplace operator

- **Taubin 1995:**

\[
\Delta_{\text{unif}} (v) := \frac{1}{|N_1(v)|} \sum_{v_i \in N_1(v)} (f(v_i) - f(v))
\]

- local geometry of the discretization (edge lengths, angles) not taken into account
- bad for non-uniform meshes

Courtesy M. Desbrun
Discrete Laplace operator

- Pinkall/Polthier 1993, Desbrun et al. 1999:

\[
\Delta_S f (v) := \frac{2}{A(v)} \sum_{v_i \in \mathcal{N}_1(v)} (\cot \alpha_i + \cot \beta_i) (f(v_i) - f(v))
\]

\[A(v) = \text{Voronoi area}\]
Discrete curvatures

- **Mean curvature:**
  \[
  \Delta_S x = -2H n
  \]

  \[
  \implies H(v) := \frac{1}{A(v)} \sum_{v_i \in N_1(v)} (\cot \alpha_i + \cot \beta_i) \| v_i - v \|
  \]

- **Gaussian curvature:**
  \[
  K(v) = \frac{1}{A(v)} \left( 2\pi - \sum_{v_i \in N_1(v)} \theta_i \right)
  \]
Discrete curvatures

**Drawback:** not intrinsic

- Same surface but $\neq$ meshes
  $\Rightarrow \neq$ curvatures

- Lots of recent work
  - See Grinspun et al. SGP 2007

- My favorite: Cohen-Steiner/Morvan 2003

Courtesy M. Meyer
Cohen-Steiner's definition

\[ \mathcal{T}(v) = \frac{1}{|B|} \sum_{\text{edges } e} \beta(e) \frac{|e \cap B|}{e} \bar{e} \bar{e}^t \]

curvature tensor
(eigenvectors = normal at v + principal curvatures)

\[ \beta(e) = \textbf{signed} \text{ angle} \]
B = approx. geodesic disk (arbitrary size)

Courtesy P. Alliez
Results

- Based on robust math background
  - Normal cycle theory
- Convergence result w.r.t. smooth surface
- Now widely used
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Definition

• Extrema of the principal curvatures along their corresponding curvature direction:

\[ e_{\text{max}} = \frac{\partial k_{\text{max}}}{\partial t_{\text{max}}} \quad e_{\text{min}} = \frac{\partial k_{\text{min}}}{\partial t_{\text{min}}} \]

\[ e_{\text{max}} = 0, \quad \partial e_{\text{max}}/\partial t_{\text{max}} < 0, \quad k_{\text{max}} > |k_{\text{min}}|, \quad \text{ridge} \]

\[ e_{\text{min}} = 0, \quad \partial e_{\text{min}}/\partial t_{\text{min}} > 0, \quad k_{\text{min}} < -|k_{\text{max}}|, \quad \text{ravine} \]

• Ridges = (convex) crest lines

Ravines = (concave) crest lines = valleys
Example

Courtesy S. Yoshizawa
Applications

- Image analysis (medical)
- Face recognition
- Shape analysis
- Compression
- Expressive rendering

Courtesy Y. Ohtake
Computation

• Difficult because:
  – Second order differential quantities
  – Need accurate estimation of principal curvatures

• Most methods need manual filtering

Courtesy S. Yoshizawa
Existing methods

- Use of (discrete) **differential operators**
  - Noisy

- Polynomial or implicit **surface fitting**
  - Need less filtering
  - Inherent smoothing difficult to control
  - Slow
  - Local vs. global approaches
Existing methods

• See works by
  – Alexander Belyaev and co-workers
    • Yutaka Ohtake SIGGRAPH 2004
    • Shin Yoshizawa SPM 2005 + Pacific Graphics 2007
  – Frédéric Cazals and Marc Pouget
    • SGP 2003, CAGD 2006
  – Klaus Hildebrandt, Konrad Polthier and Markus Wardetzky
    • SGP 2005
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Hot topics in discrete differential geometry

- **Still**: discrete Laplacian and curvatures
- **Exterior Calculus** and discrete differential forms
- **Curvature-based energies** (e.g. Willmore flow) and application to physical simulation
  - Clothes and thin plates
  - Thin shells
- **Location of streamlines and singularities**
- **Harmonic forms**
Discrete exterior calculus

- Use of (very) advanced maths to solve various geometrical problems
  - Very general but very abstract

**Keyword:** Mathieu Desbrun (Caltech)
Curvature-based energies

- Needed for (realistic) physical simulation of some special models
  - Clothes
  - Paper sheets
  - Flags
  - ...

- **Keyword:** Eitan Grinspun (Columbia Univ.)
Harmonic forms and singularities

- Useful to characterize a shape
  - More info than topology
  - Invariant under some deformations
  - Appl.: parameterization, remeshing

- Keywords:
  - Bruno Lévy (INRIA Lorraine)
  - Pierre Alliez (INRIA Sophia-Antipolis)

Courtesy P. Alliez
The end

• Next week:
  – Mesh parameterization (Cédric Gérot)
  – Point set filtering and simplification (Franck Hétroy)

• These slides will be available on the course's webpage:
  http://evasion.imag.fr/Membres/Franck.Hetroy/Teaching/Geo3D/
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Mesh smoothing papers


3. S. Fleishman et al., “Bilateral Mesh Denoising”

Mesh simplification papers

1. H. Hoppe, “Progressive Meshes” => Q. Baig

2. R. Klein et al., “Mesh Reduction with Error Control” => E. Duveau & O. Nagornaya

A Signal Processing Approach to Fair Surface Design

- Gabriel Taubin (IBM research)
- Presented at SIGGRAPH 1995
- Fairing = remove rough features (denoise, smooth)
- Main idea: surface fairing ~ signal low-pass filtering
Mathematical approach

• Fourier transform ~ Laplace transform
  - Fourier: \( F(t) = \text{cst.} \int f(w) \cdot \exp(iwt) \, dw \)
  - Laplace: \( L(t) = \int f(w) \cdot \exp(-wt) \, dw \)

• FT ~ decompose the signal into a linear combination of its Laplacian eigenvectors

• Discrete case: find a equivalent to the Laplacian operator
Discrete Laplacian

- **Discrete curve:**
  \[ \Delta x_i = \frac{1}{2}(x_{i-1} - x_i) + \frac{1}{2}(x_{i+1} - x_i) \]

- **Discrete surface:**
  \[ \Delta x_i = \sum_{j \in i^*} w_{ij} (x_j - x_i) \quad \text{with} \quad \sum_{j \in i^*} w_{ij} = 1 \]
  - Several choices proposed for the \( w_{ij} \)

- **Low-pass filtering:**
  \[ x' = f(K)x \]

K = Laplacian matrix, f = transfer function

Choice: \[ f(k) = (1 - \lambda k)(1 - \mu k) \]
Fairing algorithm

- 2 steps:
  - $x'_i = x_i + \lambda \Delta x_i$ (smoothing)
  - $x'_i = x_i + \mu \Delta x_i$ (avoid shrinkage)

\[ v_i = v_i + \{\lambda, \mu\} \sum_{j \in i^*} w_{ij} (v_j - v_i) \]
Results

- Very fast
  - $O(n)$ (also in memory)
  - FFT: $O(n \log n)$
- OK if the mesh is regular
- Not good if triangles have very different sizes/angles