Contents

Simulating Ocean Surfaces

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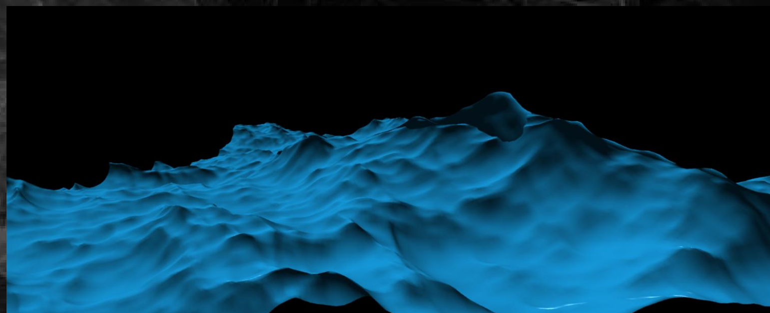
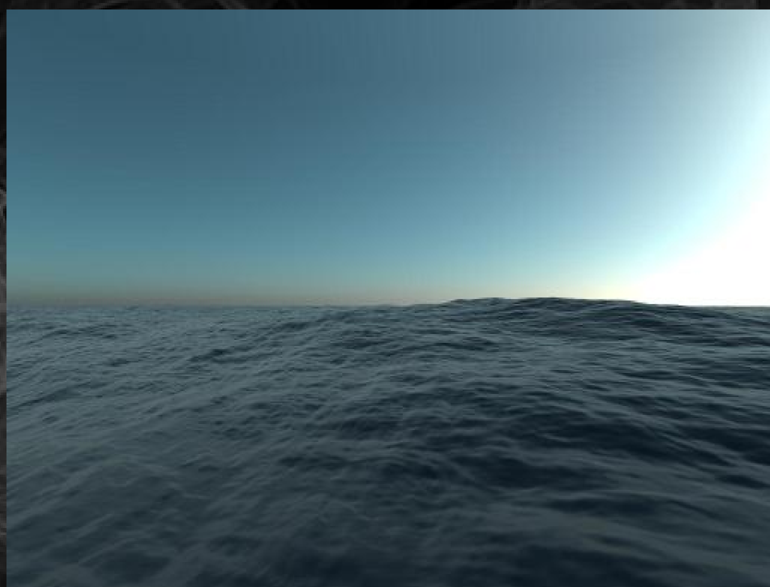
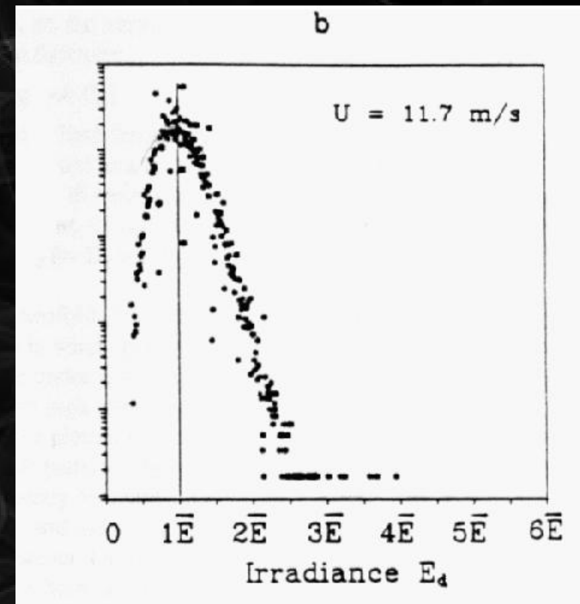
Objectives



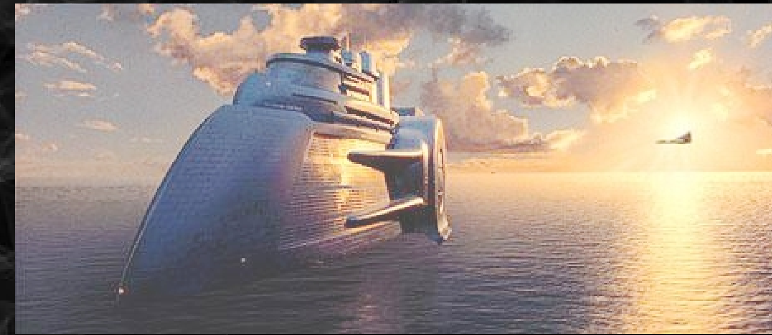




- ***Oceanography concepts***
- ***Random wave math***
- ***Hints for realistic look***
- ***Advanced things***

$$h(x,z,t) = \int_{-\infty}^{\infty} dk_x \, dk_z \, \tilde{h}(\mathbf{k},t) \exp\left\{i(k_x x + k_z z)\right\}$$

$$\tilde{h}(\mathbf{k},t) = \tilde{h}_0(\mathbf{k}) \exp\left\{-i\omega_0(\mathbf{k})t\right\} + \tilde{h}_0^*(-\mathbf{k}) \exp\left\{i\omega_0(\mathbf{k})t\right\}$$



Waterworld	13th Warrior	Fifth Element
Truman Show	Titanic	Double Jeopardy
Hard Rain	Deep Blue Sea	Devil’s Advocate
Contact	Virus	20k Leagues Under the Sea
Cast Away	World Is Not Enough	13 Days

Navier-Stokes Fluid Dynamics

Force Equation

$$\frac{\partial \mathbf{u}(\mathbf{x},t)}{\partial t} + \mathbf{u}(\mathbf{x},t)\cdot\nabla\mathbf{u}(\mathbf{x},t) + \nabla p(\mathbf{x},t)/\rho = -g\hat{\mathbf{y}} + \mathbf{F}$$

Mass Conservation

$$\nabla \cdot \mathbf{u}(\mathbf{x},t) = 0$$

Solve for functions of space and time:

- 3 velocity components
- pressure p
- density ρ distribution

Boundary conditions are important constraints

Very hard - Many scientitic careers built on this

Potential Flow

Special Substitution $\mathbf{u} = \nabla\phi(\mathbf{x}, t)$

Transforms the equations into

$$\frac{\partial\phi(\mathbf{x}, t)}{\partial t} + \frac{1}{2}\left|\nabla\phi(\mathbf{x}, t)\right|^2 + \frac{p(\mathbf{x}, t)}{\rho} + g\mathbf{x}\cdot\hat{\mathbf{y}} = 0$$

$$\nabla^2\phi(\mathbf{x}, t) = 0$$

This problem is MUCH simpler computationally and mathematically.

Free Surface Potential Flow

In the water volume, mass conservation is enforced via

$$\phi(\mathbf{x}) = \int_{\partial V} dA' \left\{ \frac{\partial \phi(\mathbf{x}')}{\partial n'} G(\mathbf{x}, \mathbf{x}') - \phi(\mathbf{x}') \frac{\partial G(\mathbf{x}, \mathbf{x}')}{\partial n'} \right\}$$

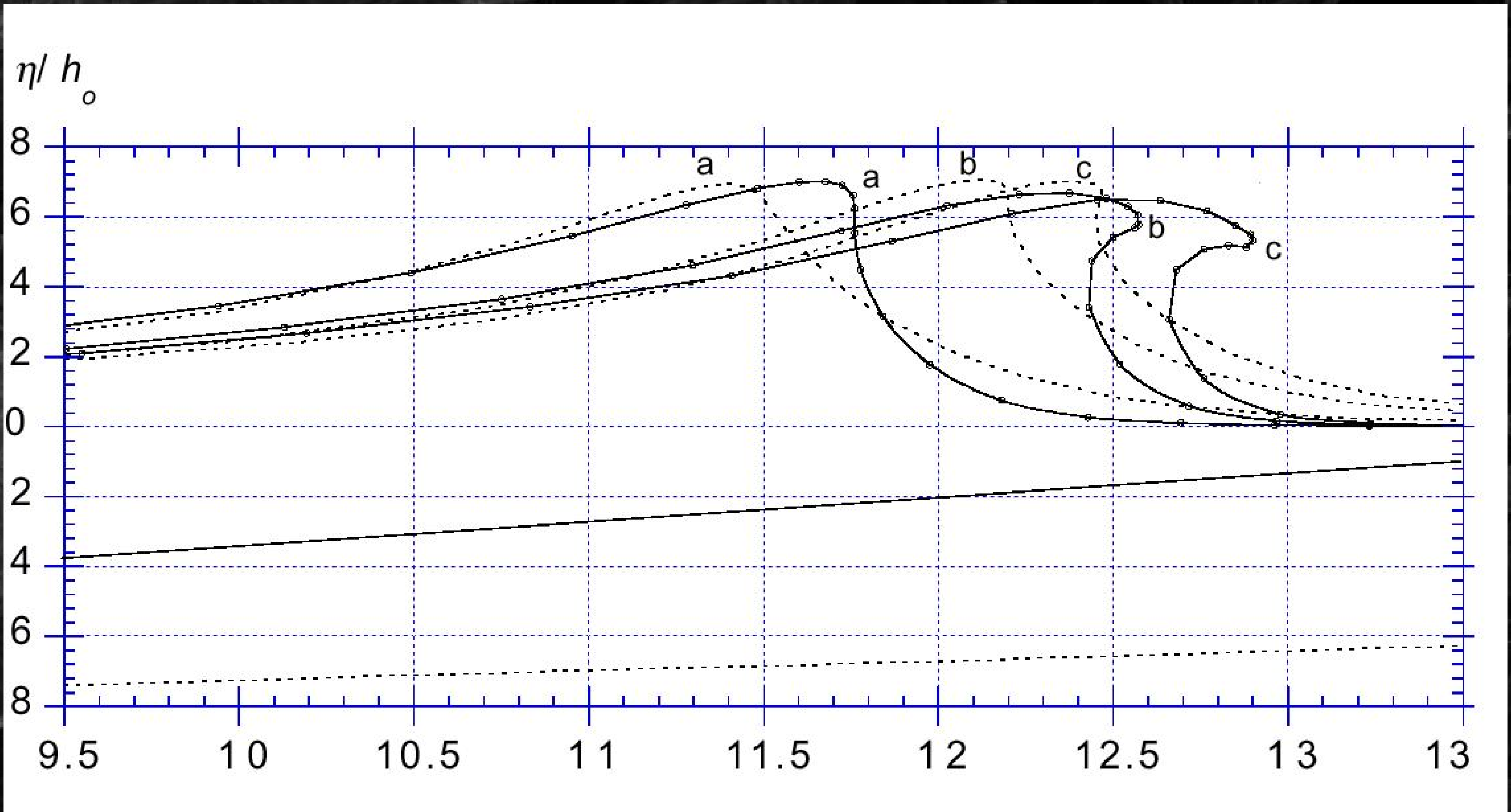
At points $\mathbf{r}$ on the surface

$$\frac{\partial \phi(\mathbf{r}, t)}{\partial t} + \frac{1}{2} \left| \nabla \phi(\mathbf{r}, t) \right|^2 + \frac{p(\mathbf{r}, t)}{\rho} + g\mathbf{r} \cdot \hat{\mathbf{y}} = 0$$

Dynamics of surface points:

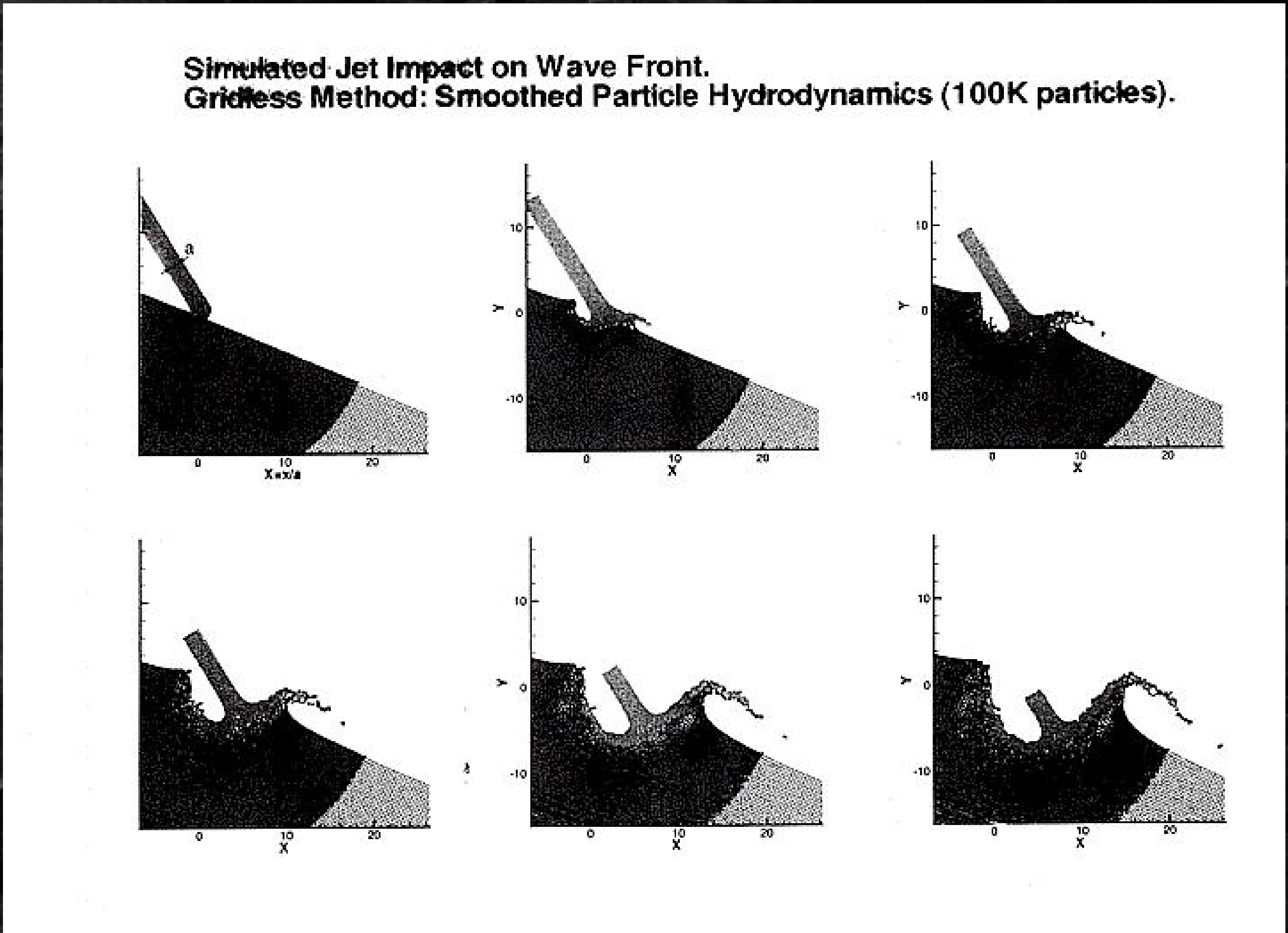
$$\frac{d\mathbf{r}(t)}{dt} = \nabla \phi(\mathbf{r}, t)$$

Numerical Wave Tank Simulation



Grilli, Guyenne, Dias (2000)

Plunging Break and Splash Simulation



Tulin (1999)

Simplifying the Problem

Road to practicality - ocean surface:

- Simplify equations for relatively mild conditions
- Fill in gaps with oceanography.

Original dynamical equation at 3D points in volume

$$\frac{\partial \phi(\mathbf{r},t)}{\partial t} + \frac{1}{2}\left|\nabla \phi(\mathbf{r},t)\right|^2 + \frac{p(\mathbf{r},t)}{\rho} + g\mathbf{r}\cdot\hat{\mathbf{y}} = 0$$

Equation at 2D points (x,z) on surface with height h

$$\frac{\partial \phi(x,z,t)}{\partial t} = -gh(x,z,t)$$

Simplifying the Problem: Mass Conservation

Vertical component of velocity

$$\frac{\partial h(x,z,t)}{\partial t} = \hat{\mathbf{y}} \cdot \nabla \phi(x,z,t)$$

Use mass conservation condition

$$\hat{\mathbf{y}} \cdot \nabla \phi(x,z,t) \sim \left(\sqrt{-\nabla_H^2}\right)\phi = \left(\sqrt{-\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial z^2}}\right)\phi$$

Linearized Surface Waves

$$\frac{\partial h(x,z,t)}{\partial t} = \left(\sqrt{-\nabla_H^2}\right)\phi(x,z,t)$$

$$\frac{\partial \phi(x,z,t)}{\partial t} = -gh(x,z,t)$$

General solution easily computed in terms of Fourier Transforms

Solution for Linearized Surface Waves

General solution in terms of Fourier Transform

$$h(x, z, t) = \int_{-\infty}^{\infty} dk_x \, dk_z \, \tilde{h}(\mathbf{k}, t) \; \exp\left\{i(k_x x + k_z z)\right\}$$

with the amplitude depending on the *dispersion relationship*

$$\omega_0(\mathbf{k}) = \sqrt{g\,|\mathbf{k}|}$$

$$\tilde{h}(\mathbf{k}, t) = \tilde{h}_0(\mathbf{k}) \exp\left\{-i\omega_0(\mathbf{k})t\right\} + \tilde{h}_0^*(-\mathbf{k}) \exp\left\{i\omega_0(\mathbf{k})t\right\}$$

The complex amplitude $\tilde{h}_0(\mathbf{k})$ is arbitrary.

Oceanography

- Think of the heights of the waves as a kind of random process
- Decades of detailed measurements support a statistical description of ocean waves.
- The wave height has a spectrum

$$\left\langle \left| \tilde{h}_0(\mathbf{k}) \right|^2 \right\rangle = P_0(\mathbf{k})$$

- Oceanographic models tie P_0 to environmental parameters like wind velocity, temperature, salinity, etc.

Models of Spectrum

- Wind speed V
- Wind direction vector $\hat{\mathrm{V}}$ (horizontal only)
- Length scale of biggest waves $L = V^2/g$ (g=gravitational constant)

Phillips Spectrum

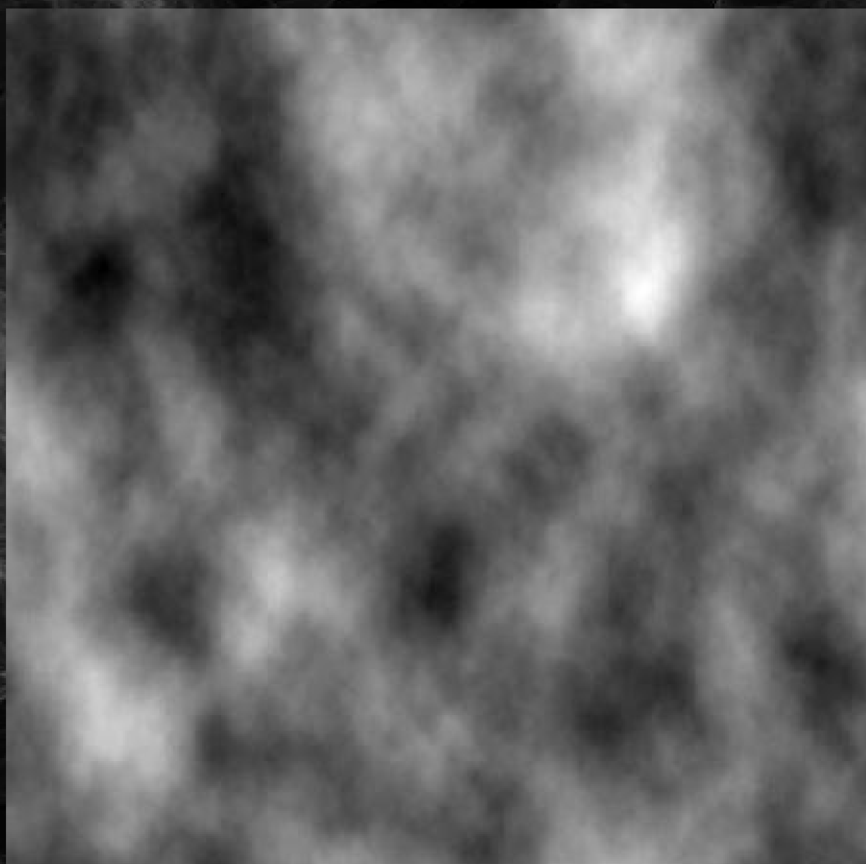
$$P_0(\mathbf{k}) = \left|\hat{\mathbf{k}} \cdot \hat{\mathbf{V}}\right|^2 \frac{\exp(-1/k^2L^2)}{k^4}$$

JONSWAP Frequency Spectrum

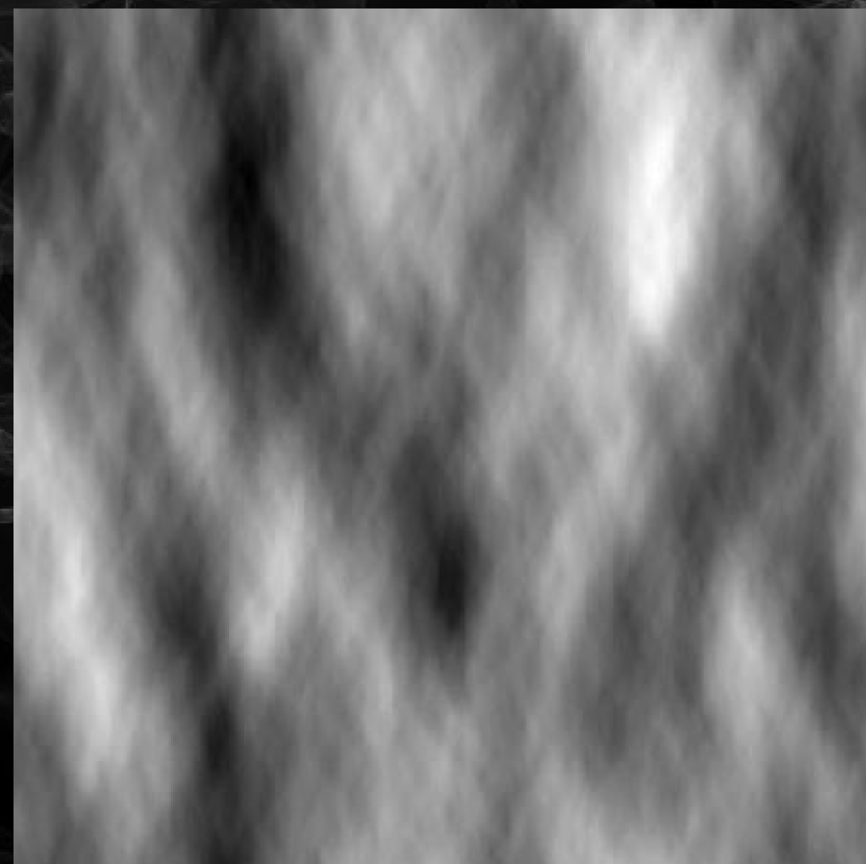
$$P_0(\omega) = \frac{\exp\left\{-\frac{5}{4}\left(\frac{\omega}{\Omega}\right)^{-4} + e^{-(\omega-\Omega)^2/2(\sigma\Omega)^2}\ln\gamma\right\}}{\omega^5}$$

Variation in Wave Height Field

Pure Phillips Spectrum



Modified Phillips Spectrum



Simulation of a Random Surface

Generate a set of “random” amplitudes on a grid

$$\tilde{h}_0(\mathbf{k}) = \xi e^{i\theta}\sqrt{P_0(\mathbf{k})}$$

ξ = Gaussian random number, mean 0 & std dev 1

θ = Uniform random number [0,2π].

$$k_x = \frac{2\pi}{\Delta x}\frac{n}{N} \quad (n = -N/2, \ldots, (N-1)/2)$$

$$k_z = \frac{2\pi}{\Delta z}\frac{m}{M} \quad (m = -M/2, \ldots, (M-1)/2)$$

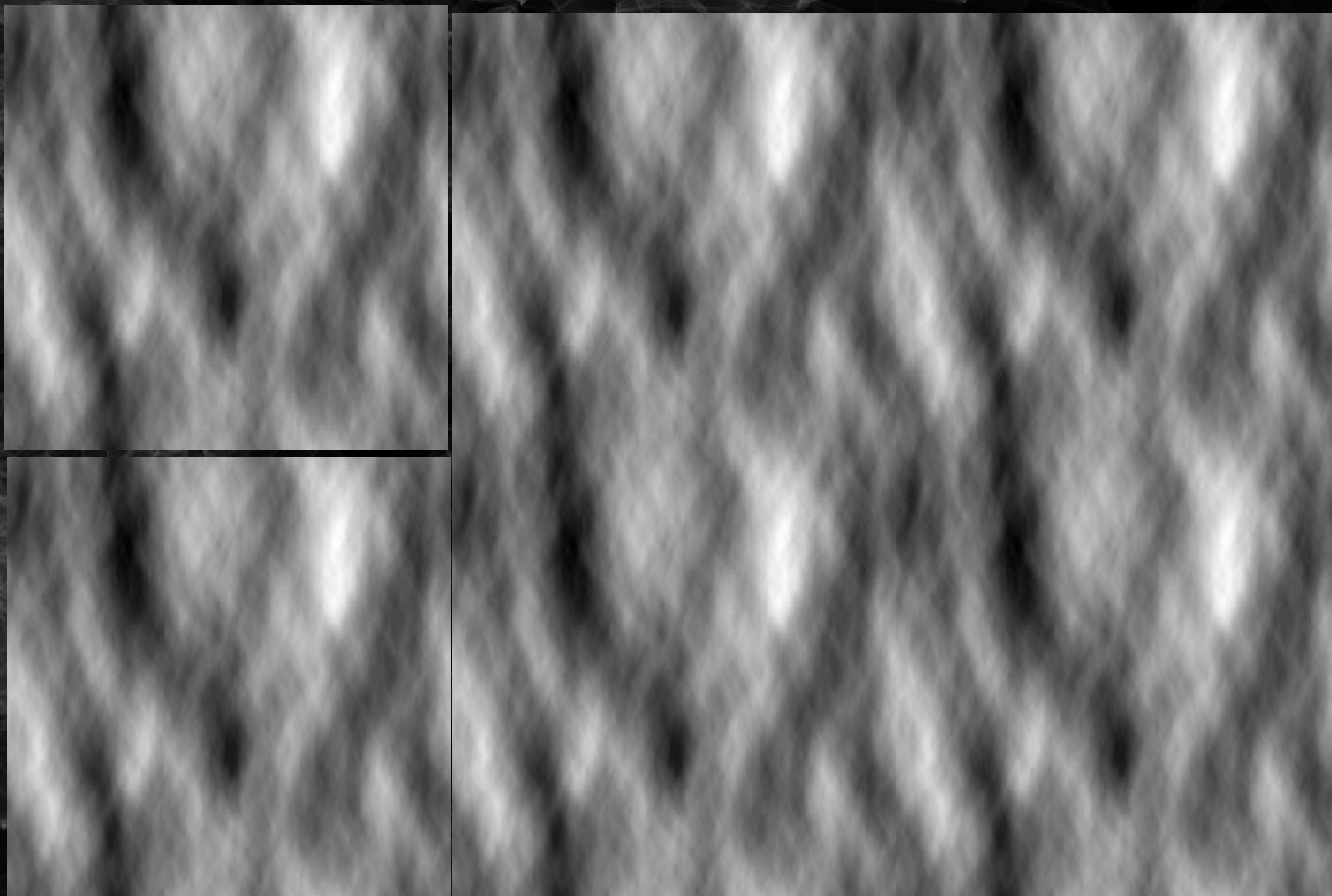
FFT of Random Amplitudes

Use the Fast Fourier Transform (FFT) on the amplitudes to obtain the wave height realization $h(x, z, t)$

Wave height realization exists on a regular, periodic grid of points.

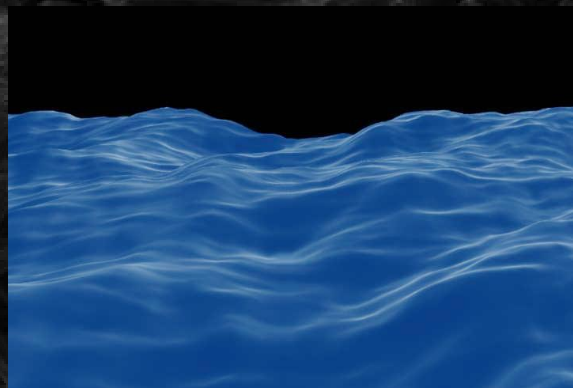
$$x = n\Delta x \quad (n = -N/2, \ldots, (N-1)/2)$$
$$z = m\Delta z \quad (m = -M/2, \ldots, (M-1)/2)$$

The realization tiles seamlessly. This can sometimes show up as repetitive waves in a render.

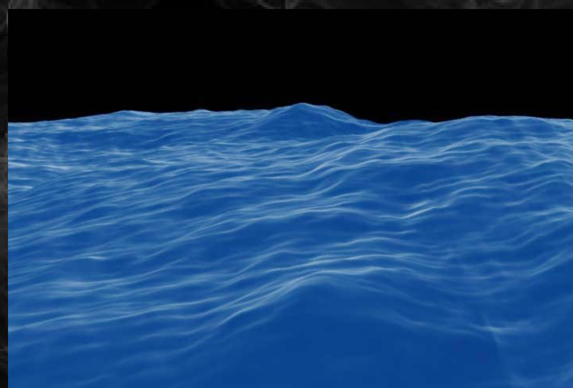


High Resolution Rendering
Sky reflection, upwelling light, sun glitter
1 inch facets, 1 kilometer range

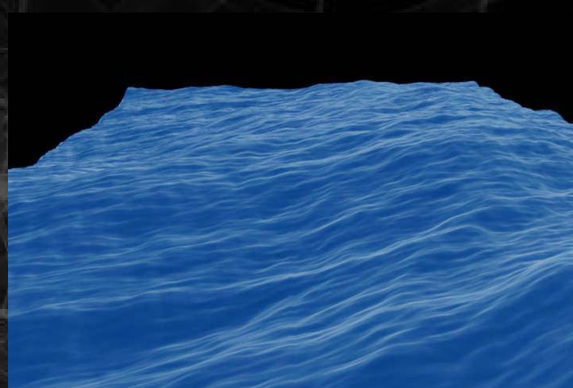
Effect of Resolution



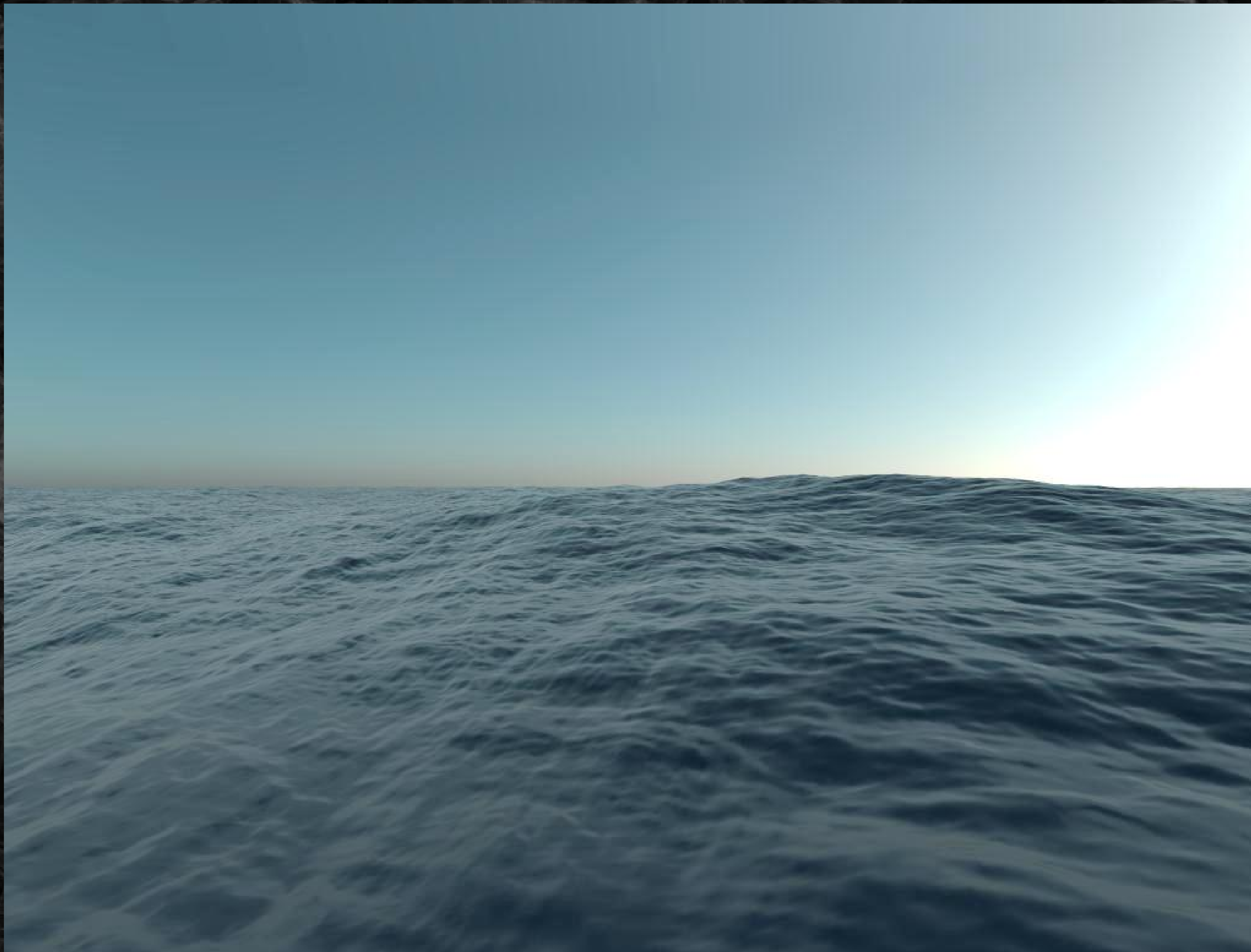
Low : 100 cm facets



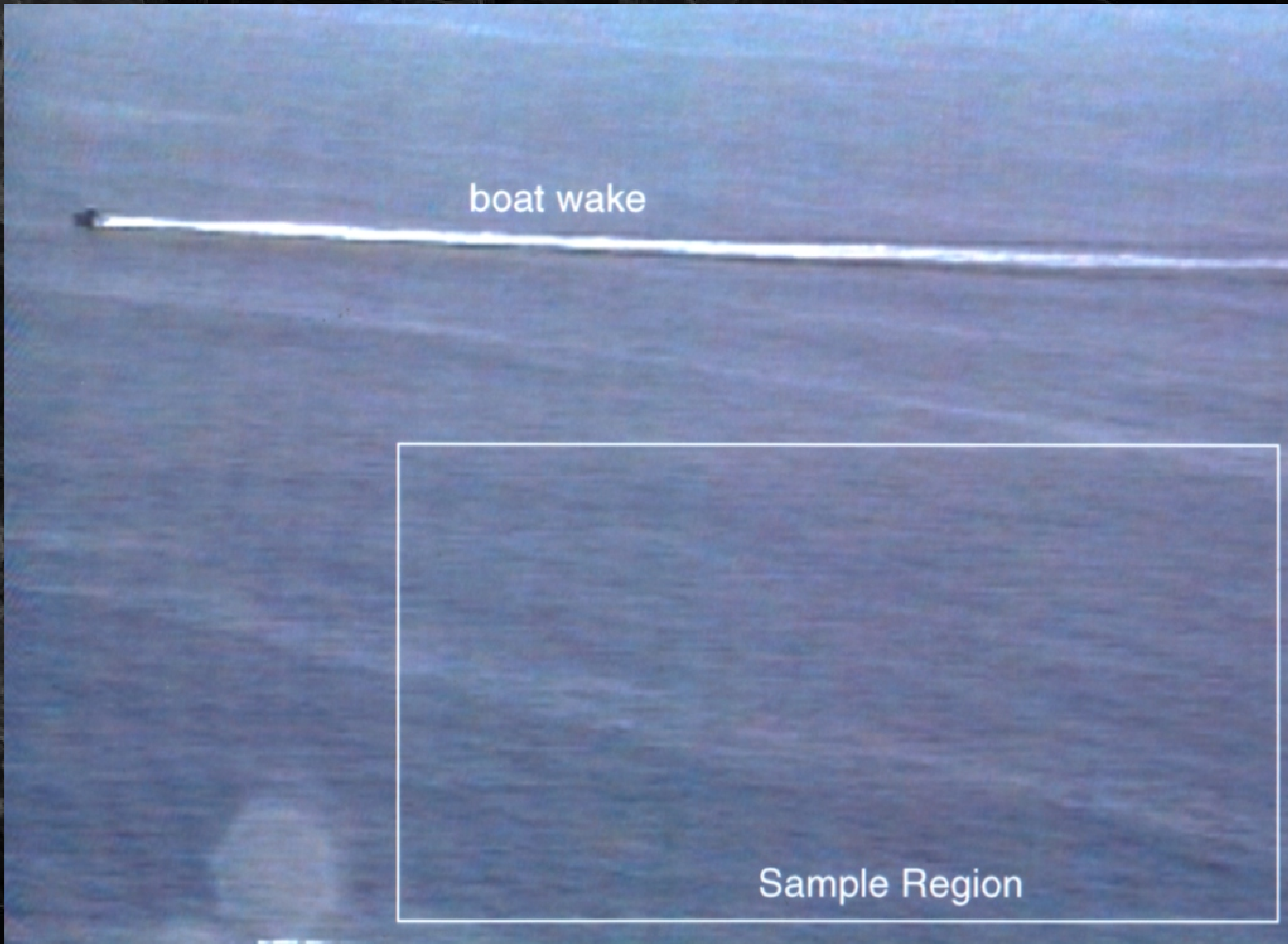
Medium : 10 cm facets



High : 1 cm facets



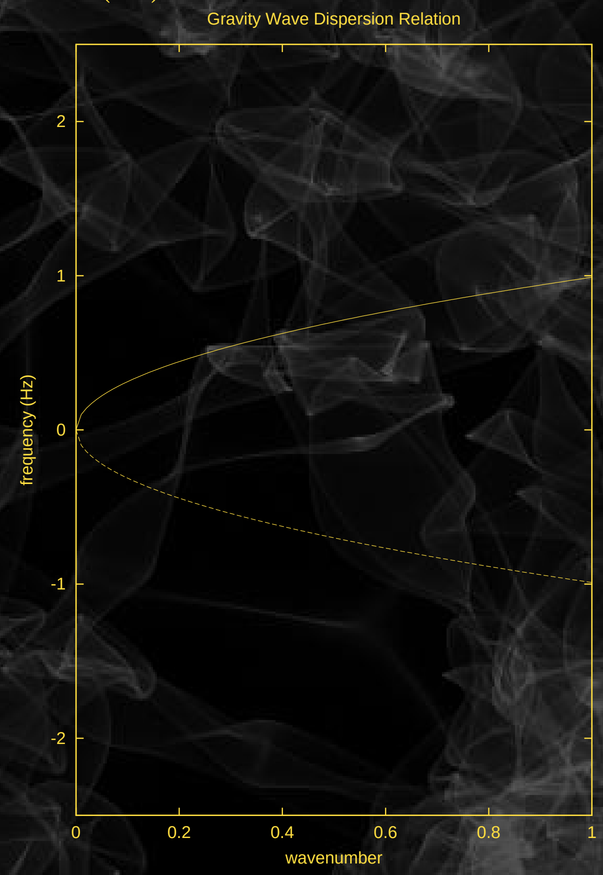
Simple Demonstration of Dispersion



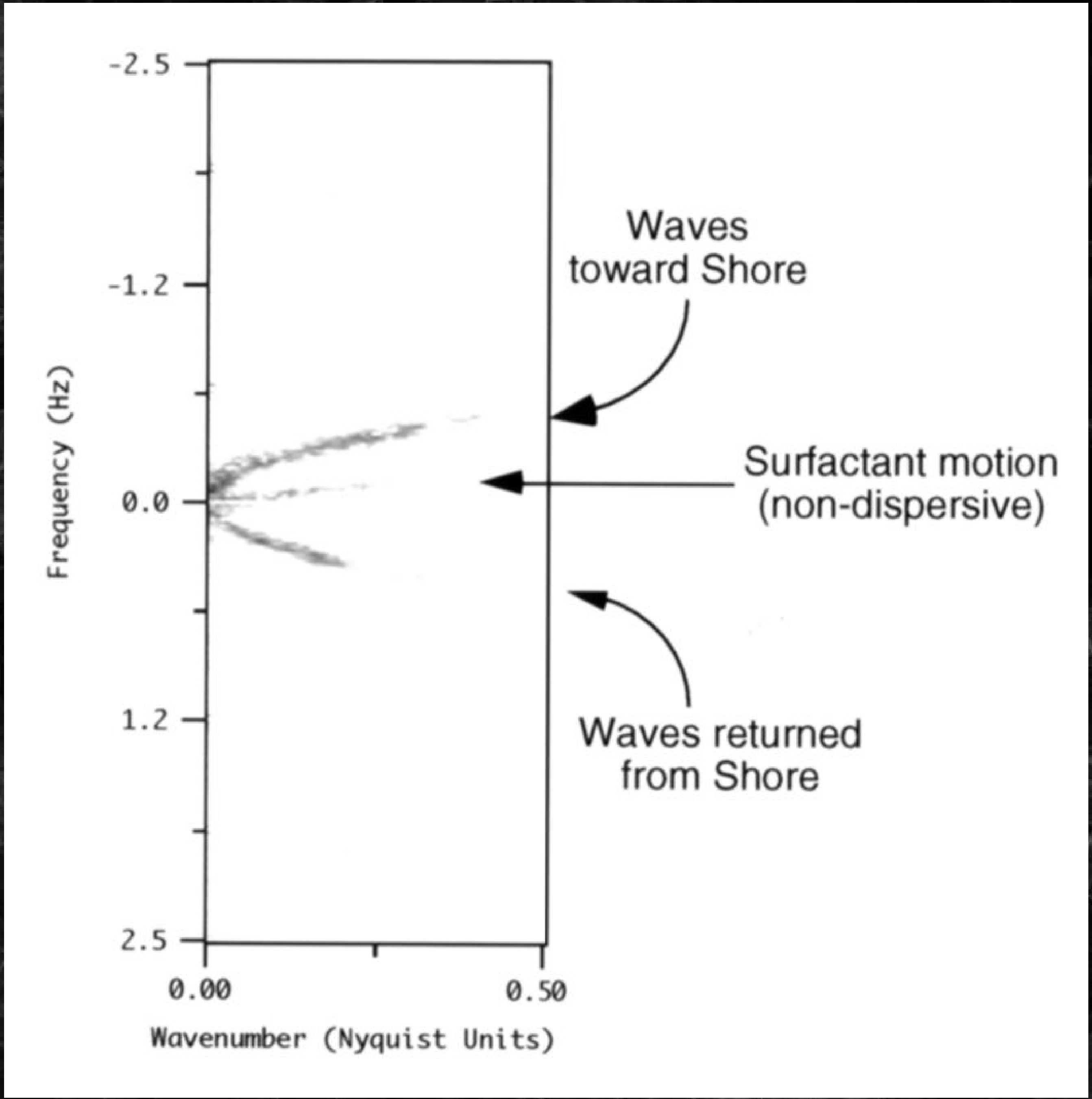
256 frames, 256×128 region

Data Processing

- Fourier transform in both time and space: $\tilde{h}(\mathbf{k}, \omega)$
- Form Power Spectral Density $P(\mathbf{k}, \omega) = \left\langle \left| \tilde{h}(\mathbf{k}, \omega) \right|^2 \right\rangle$
- If the waves follow dispersion relationship, then P is strongest at frequencies $\omega = \omega(k)$.



Processing Results



Looping in Time – Continuous Loops

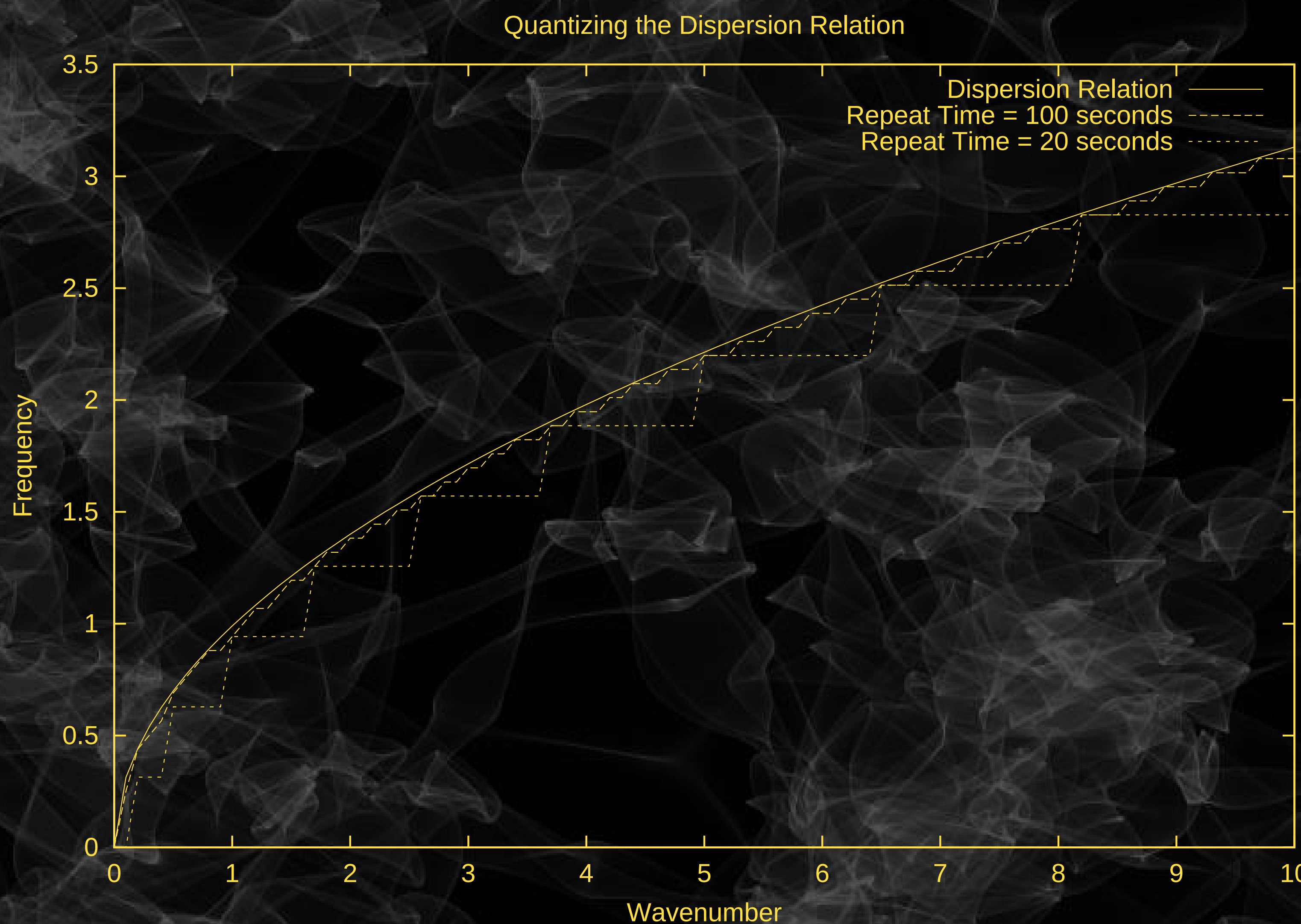
- Continuous loops can't be made because dispersion doesn't have a fundamental frequency.
- Loops can be made by modifying the dispersion relationship.

Repeat time T

Fundamental Frequency $\omega_0 = \frac{2\pi}{T}$

New dispersion relation $\tilde{\omega} = \text{integer}\left(\frac{\omega(k)}{\omega_0}\right) \ \omega_0$

Quantized Dispersion Relation



Hamiltonian Approach for Surface Waves

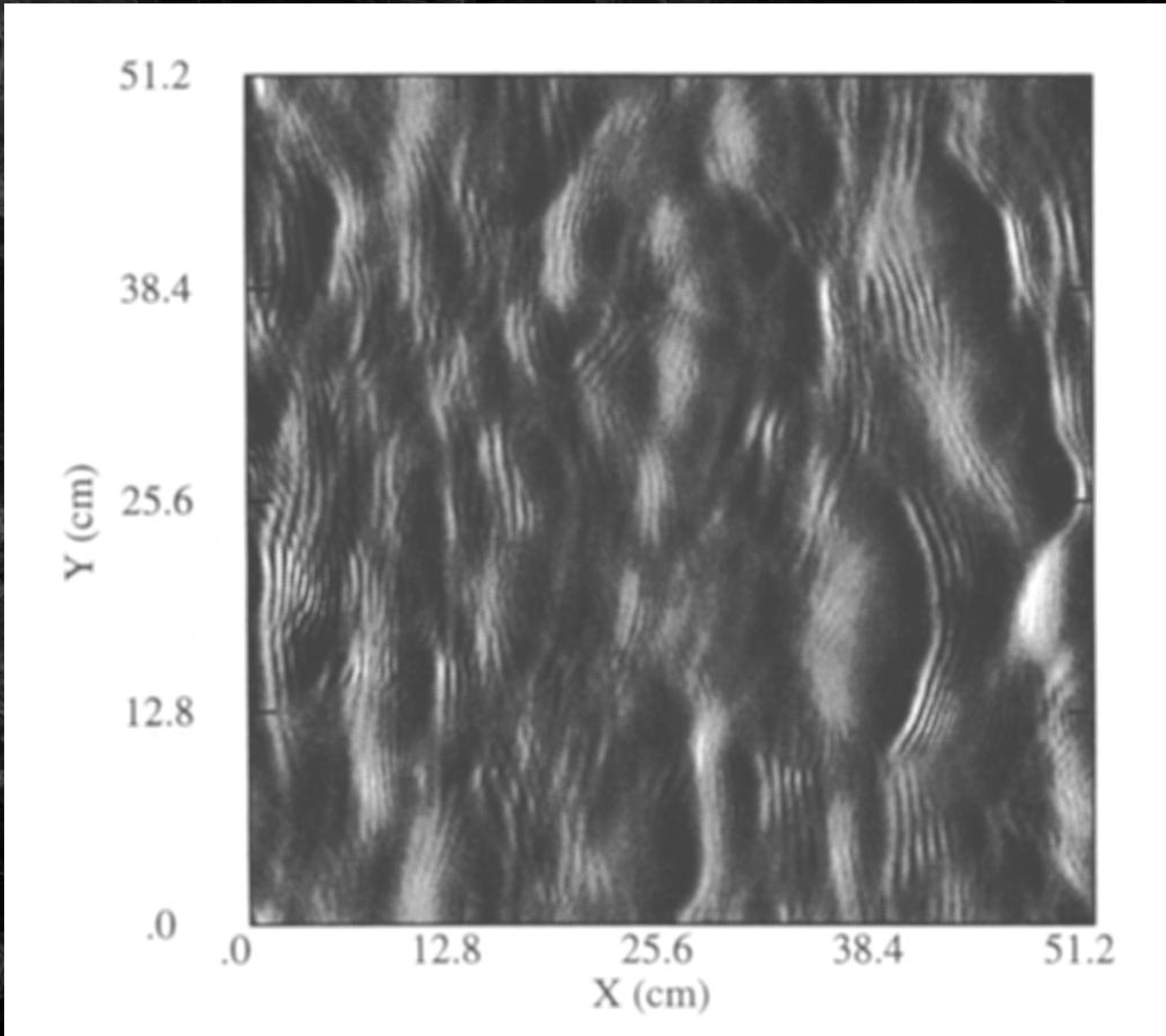
Miles, Milder, Henyey, ...

- If a crazy-looking surface operator like $\sqrt{-\nabla_H^2}$ is ok, the exact problem can be recast as a *canonical problem* with momentum ϕ and coordinate h in 2D.

- Milder has demonstrated numerically:
  – The onset of wave breaking
  – Accurate capillary wave interaction

- Henyey *et al.* introduced *Canonical Lie Transformations*:
  – Start with the solution of the linearized problem - (ϕ_0, h_0)
  – Introduce a continuous set of transformed fields - (ϕ_q, h_q)
  – The exact solution for surface waves is for $q = 1$.

Surface Wave Simulation (Milder, 1990)



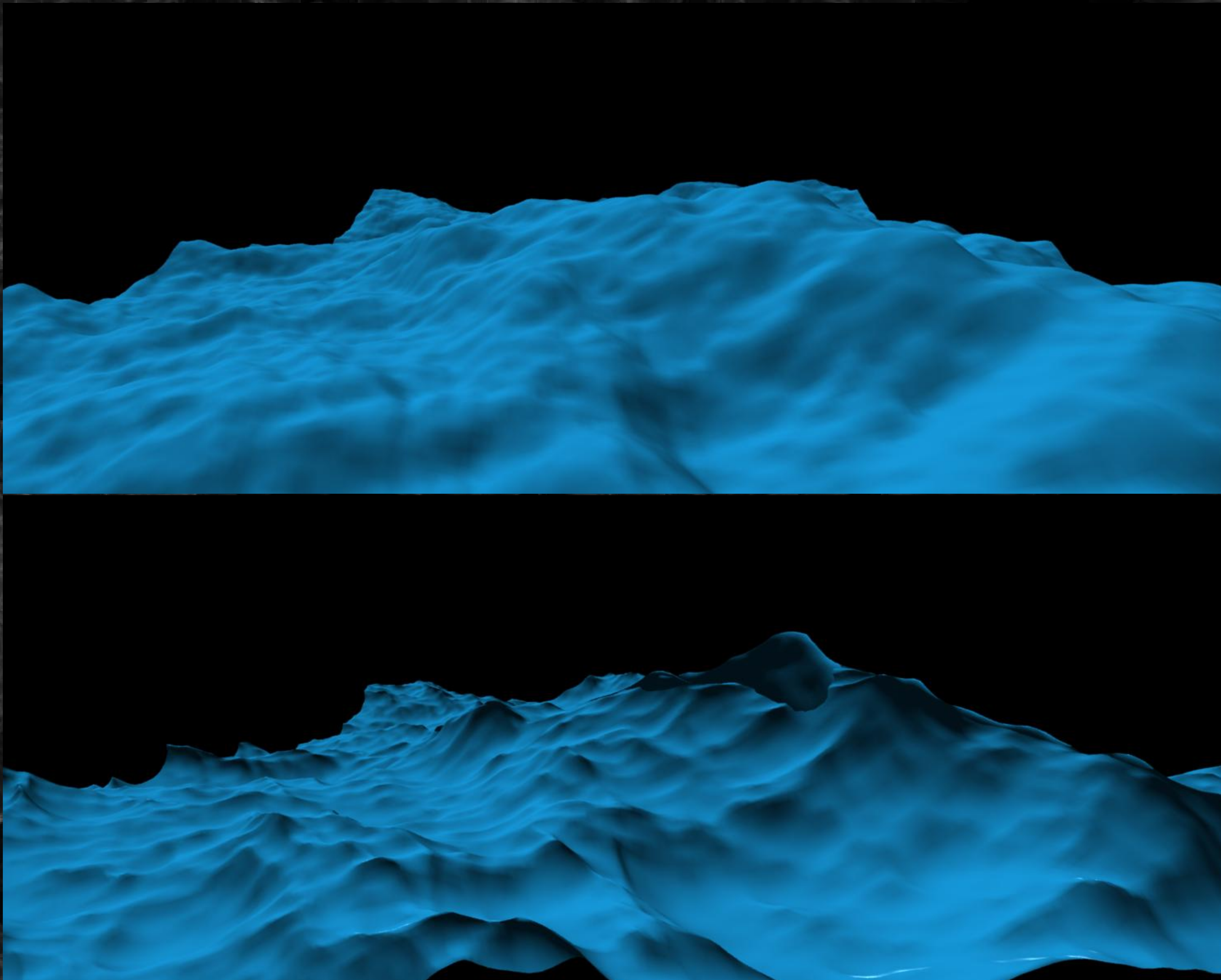
Choppy, Near-Breaking Waves

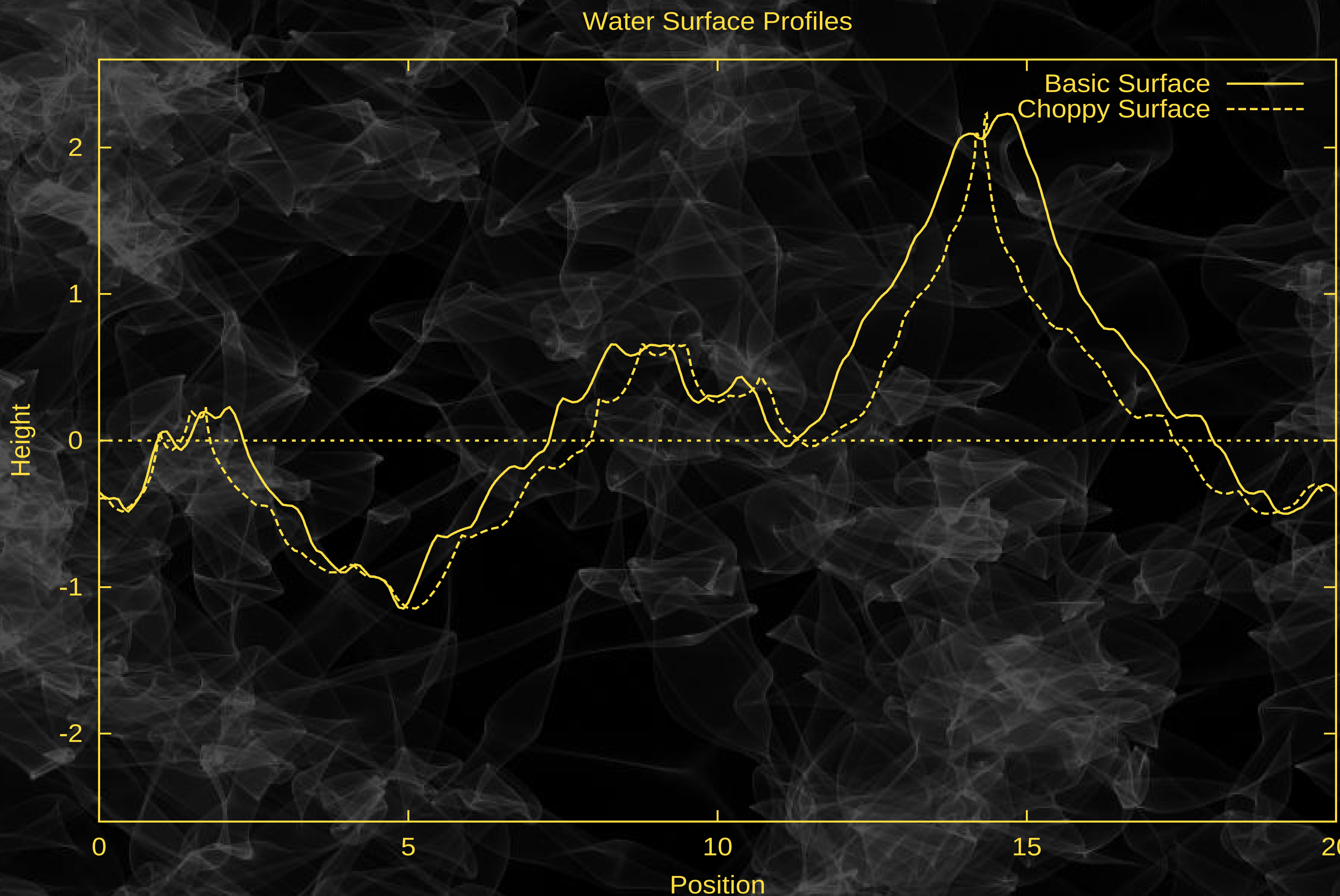
Horizontal velocity becomes important for distorting wave.

Wave at $\mathbf{x}$ morphs horizontally to the position $\mathbf{x} + \mathbf{D}(\mathbf{x}, t)$

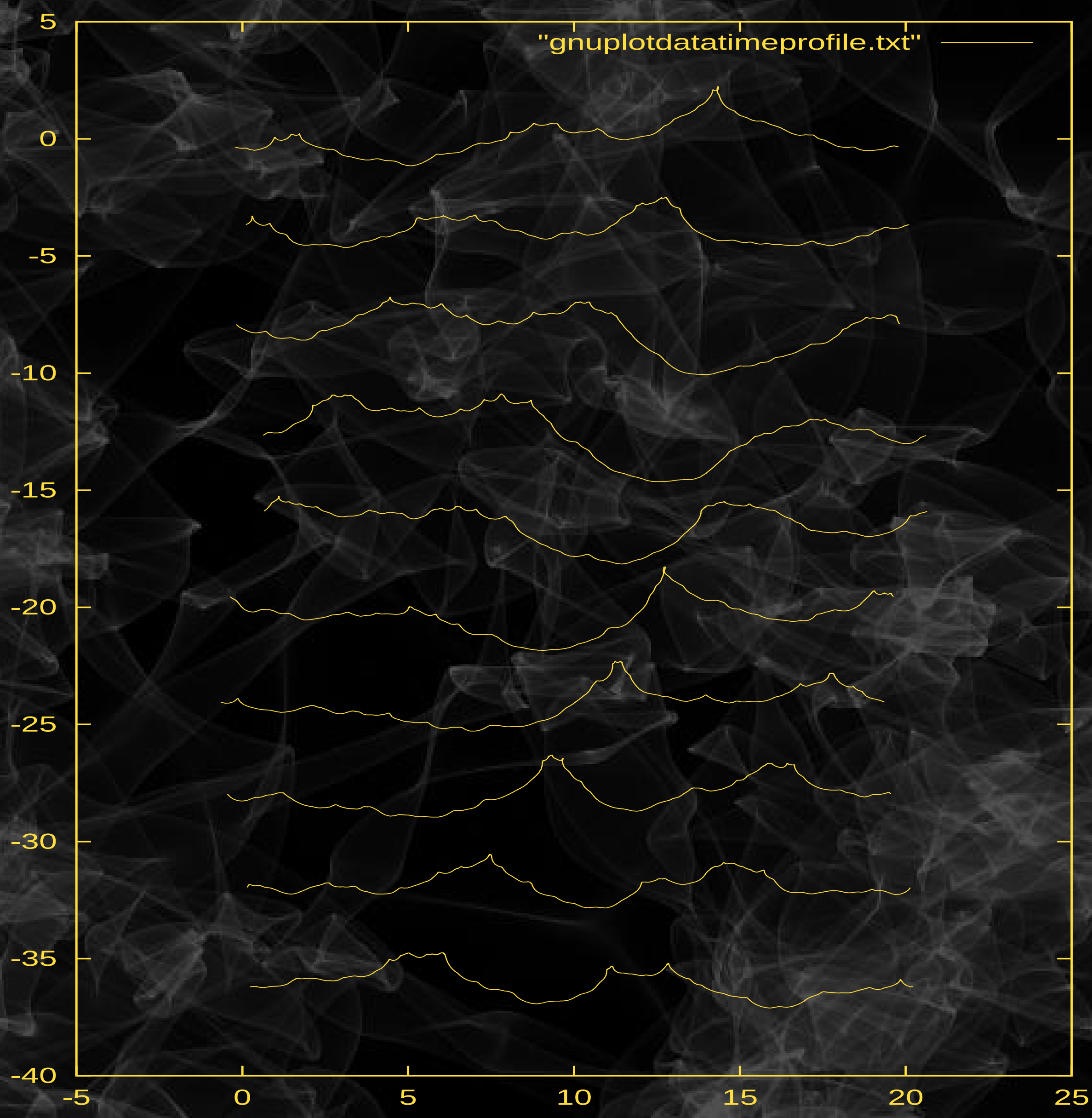
$$\mathbf{D}(\mathbf{x}, t) = -\lambda \int d^2k \, \frac{i\mathbf{k}}{|\mathbf{k}|} \tilde{h}(\mathbf{k}, t) \, \exp\{i(k_x x + k_z z)\}$$

The factor λ allows artistic control over the magnitude of the morph.




Water Surface Profiles
Basic Surface
Choppy Surface
Height
Position
2
1
0
-1
-2
0
5
10
15
20

Time Sequence of Choppy Waves



Choppy Waves: Detecting Overlap

$$\mathbf{x} \rightarrow \mathbf{X}(\mathbf{x}, t) = \mathbf{x} + \mathbf{D}(\mathbf{x}, t)$$

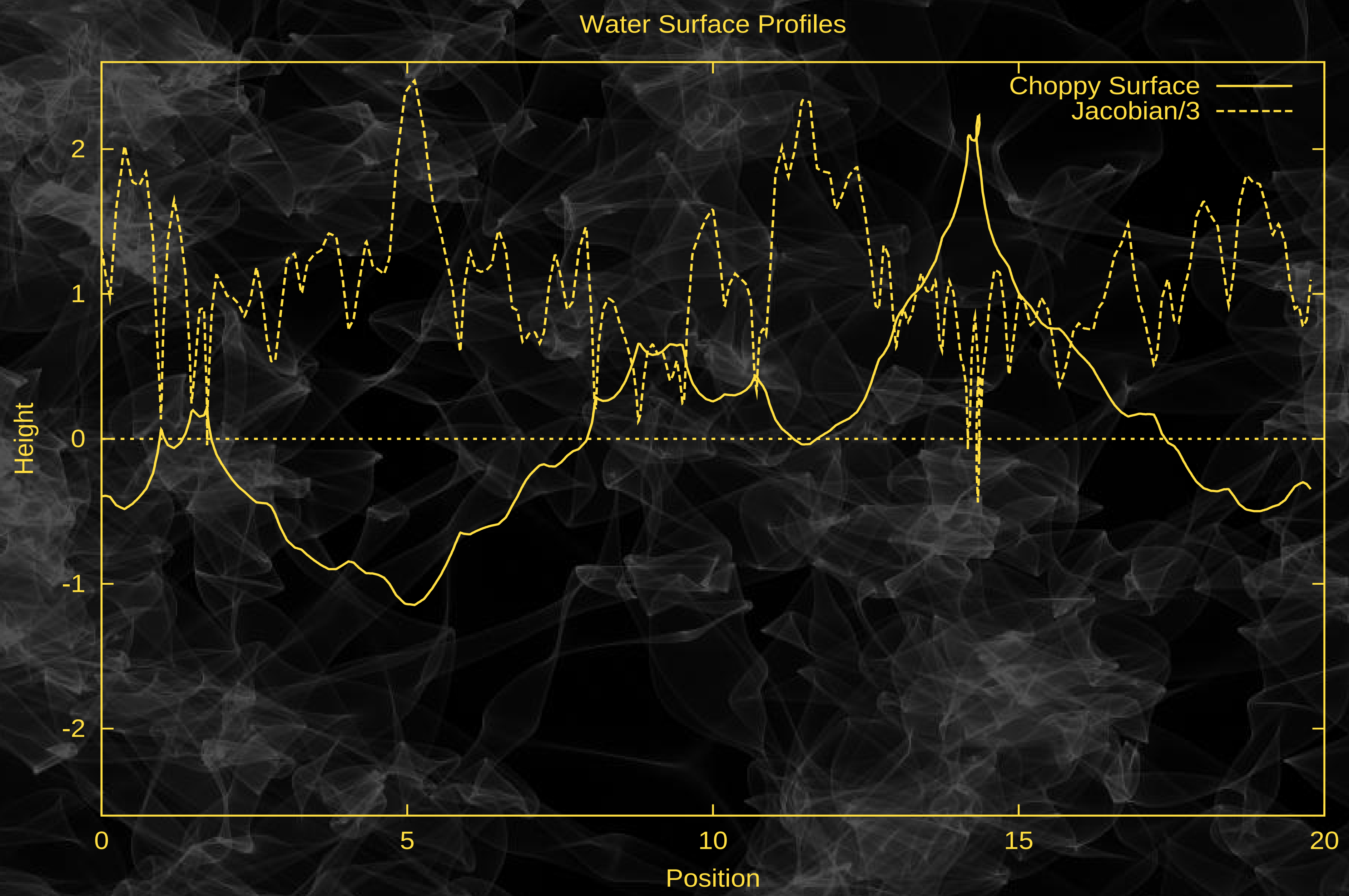
is unique and invertible as long as the surface does not intersect itself.

When the mapping intersects itself, it is not unique. The quantitative measure of this is the *Jacobian* matrix

$$J(\mathbf{x}, t) = \begin{bmatrix} \partial \mathbf{X}_x / \partial x & \partial \mathbf{X}_x / \partial z \\ \partial \mathbf{X}_z / \partial x & \partial \mathbf{X}_z / \partial z \end{bmatrix}$$

The signal that the surface intersects itself is

$$\det(J) \leq 0$$



Water Surface Profiles
Choppy Surface
Jacobian/3
Height
Position
2
1
0
-1
-2
0
5
10
15
20

Learning More About Overlap

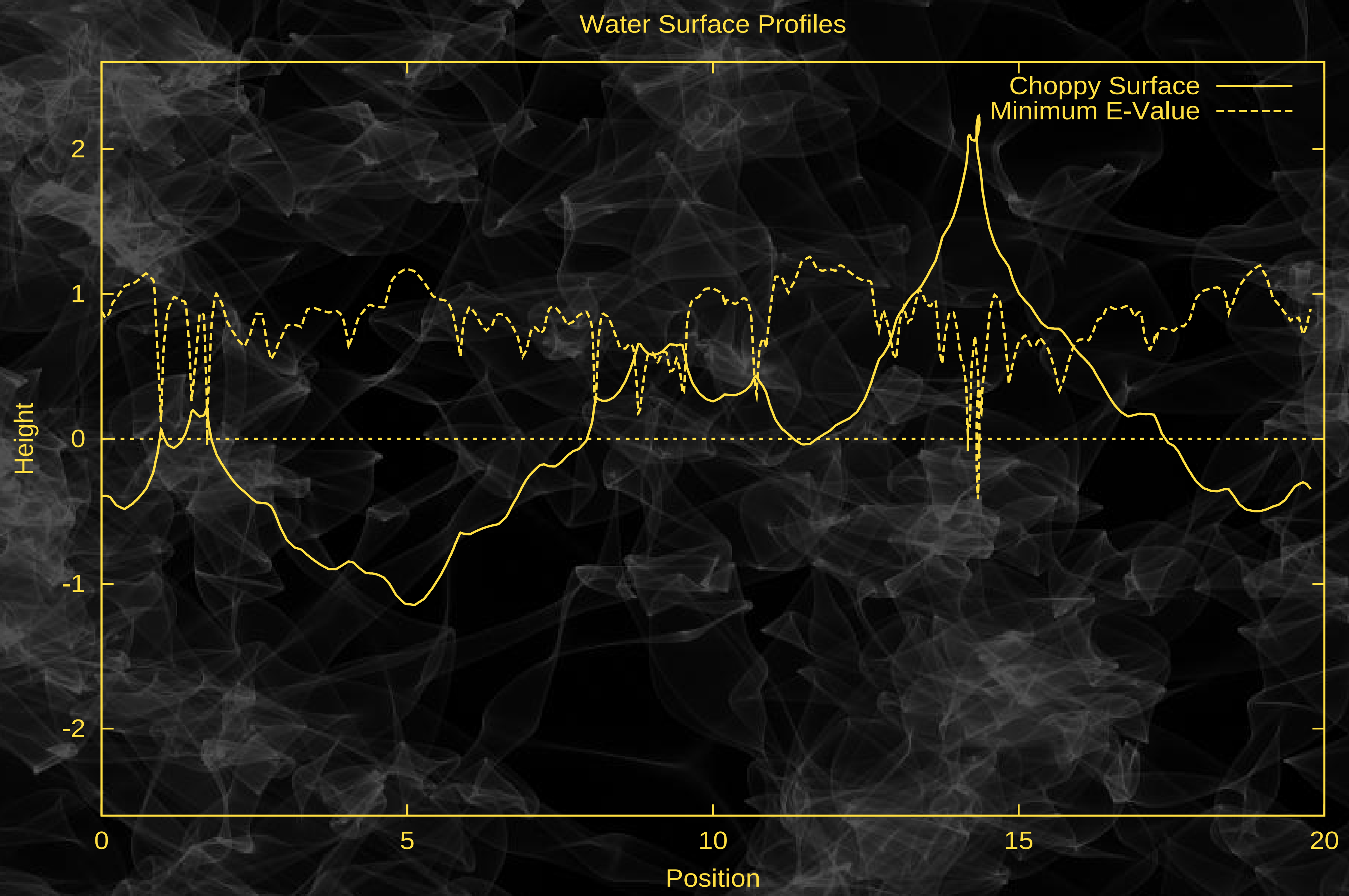
Two *eigenvalues*, $J_- \leq J_+$, and *eigenvectors* $\hat{\mathrm{e}}_-$, $\hat{\mathrm{e}}_+$

$$J = J_- \hat{\mathrm{e}}_- \hat{\mathrm{e}}_- + J_+ \hat{\mathrm{e}}_+ \hat{\mathrm{e}}_+$$

$$\det(J) = J_- J_+$$

For no chop, $J_- = J_+ = 1$. As the displacement magnitude increases, J_+ stays positive while J_- becomes negative at the location of overlap.

At overlap, $J_- < 0$, the alignment of the overlap is parallel to the eigenvalue $\hat{\mathrm{e}}_-$.


Water Surface Profiles
Choppy Surface
Minimum E-Value
Height
Position
2
1
0
-1
-2
0
5
10
15
20

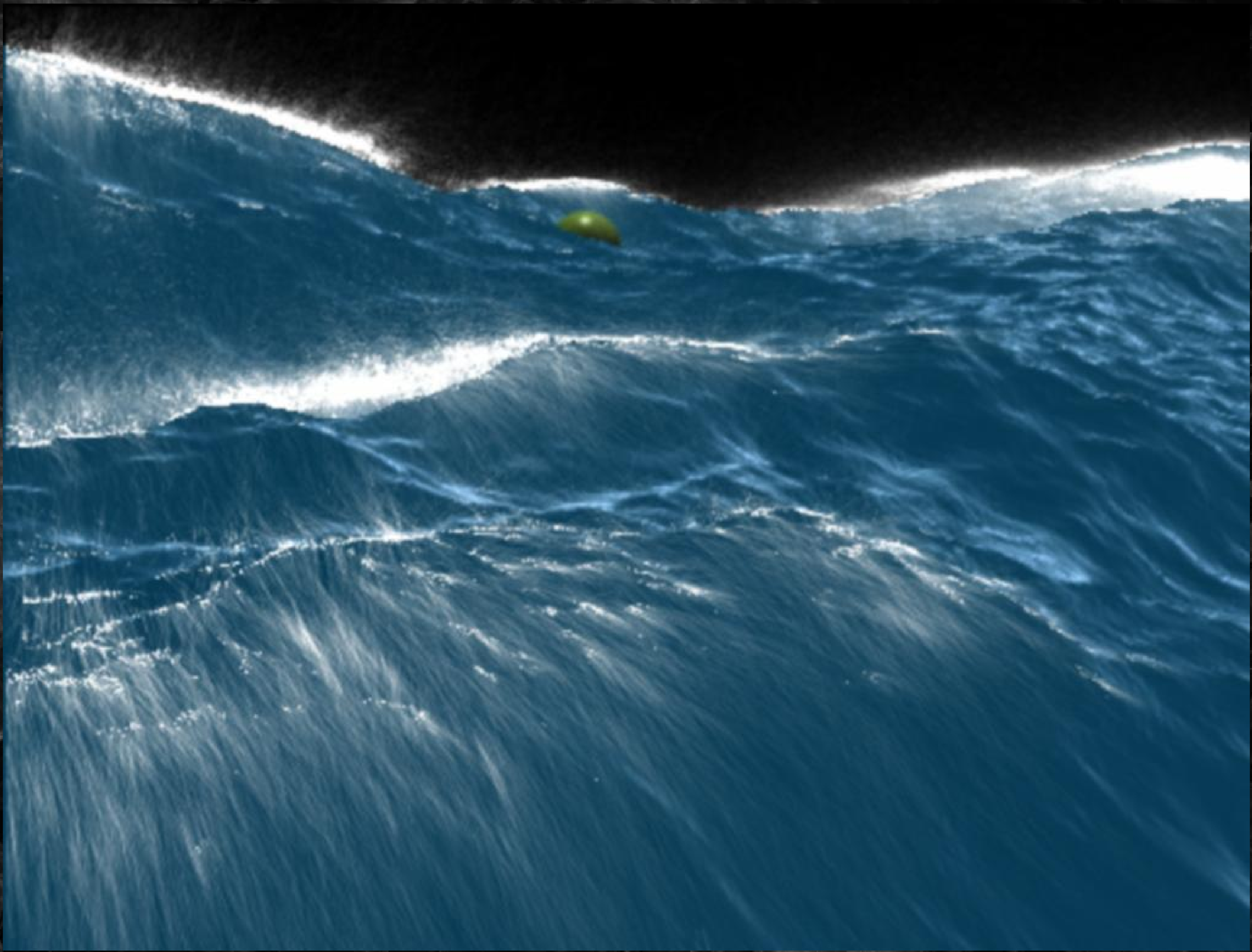
Simple Spray Algorithm

- Pick a point on the surface at random
- Emit a spray particle if $J_- < J_T$ threshold
- Particle initial direction ($\hat{\mathbf{n}}$ = surface normal)

$$\hat{\mathbf{v}} = \frac{(J_T - J_-)\hat{\mathbf{e}}_- + \hat{\mathbf{n}}}{\sqrt{1 + (J_T - J_-)^2}} \tag{1}$$

- Particle initial speed from a half-gaussian distribution with mean proportional to $J_T - J_-$.
- Simple particle dynamics: gravity and wind drag

Surface and Spray Render



Summary

- FFT-based random ocean surfaces are fast to build, realistic, and flexible.
- Based on a mixture of theory and experimental phenomenology.
- Used alot in professional productions.
- Real-time capable for games
- Lots of room for more complex behaviors.

Latest version of course notes and slides:

http://home1.gte.net/tssndrf/index.html

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